Functional analysis: Problem set 1

F

Problem 1: X is a Banach Space and $A \subseteq X$ is a nonempty subset. Assume that $\frac{x+y}{2} \in A$ whenever $x, y \in A$. Show that A is convex. (FALSE!)

1)

Problem 2: $f: \mathbb{R}^d \to \mathbb{R}$ is a convex function. Show that $A_f := \{(x, y) \in \mathbb{R}^d \times \mathbb{R} \mid y \geq f(x)\}$ is a convex set in $\mathbb{R}^d$.

Conversely, if A_f is a convex set, show that f is a convex function.

Problem 3: Show that the following sets are convex but not ~~totally~~ ~~conve~~ perfectly conve:

(1) $X = \ell^2 = \{(x_1, x_2, \dots) \mid x_i \in \mathbb{R}, \sum_{i=1}^{\infty} x_i^2 < \infty\}$, $A = \{x \in \ell^2 \mid x_i = 0 \text{ for all but finitely many } i\}$

(2) $X = M(\mathbb{R})$ = space of all regular Borel measures on $\mathbb{R}$,
$A = \{\mu \in X \mid \mu \text{ is supported on finitely many points}\}$

(3) $X = C[0,1]$, $A = \{f \in C[0,1] \mid f \in C^1\}$ (continuously differentiable functions)

2)

Problem 4: A is a nonempty, closed convex subset of a Banach space X.
Show that A is perfectly convex.

Problem 5: W is a subspace of a Banach space X. Show that

(a) W is convex

(b) W is perfectly convex if and only if W is closed.

Problem 6: A is a convex set in X (a Banach space). Suppose $x \in \check{A}$ and $y \in A$.
Show that $\alpha x + (1-\alpha) y \in \check{A}$ $\forall \alpha \in (0, 1]$.

Problem 7: $T: X \to Y$ is a bounded linear transformation (X, Y are Banach spaces).

(a) If $A \subseteq X$ is perfectly convex and bounded, show that $T(A)$ is perfectly convex

(b) If $B \subseteq X$ is perfectly convex, show that $T^{-1}(B)$ is perfectly convex

Functional analysis. Problem set 2:

1) In this exercise you are guided to a proof of $\mathring{\bar{A}} \subseteq \mathring{A}$ for a perfectly convex set $A \subseteq X$

(a) Assume that $0 \in \mathring{\bar{A}}$. Show that $\exists\, \epsilon > 0$ such that $B_0(\epsilon) \subseteq \overline{(A \cap B_0(\epsilon))} + \frac{1}{2} B_0(\epsilon)$

[Here the notation $C+D$ means $\{x+y \mid x \in C, y \in D\}$]

(b) Show that for ϵ as above, $\frac{1}{2^m} B_0(\epsilon) \subseteq \frac{1}{2^m}\overline{(A \cap B_0(\epsilon))} + \frac{1}{2^{m+1}} B_0(\epsilon)$ $\forall m = 1, 2, 3, \ldots$

(c) Given $x \in \frac{1}{2} B_0(\epsilon)$, show that $\exists\, y_1, y_2, \ldots$ such that [illegible] $x = \sum_{n=1}^{\infty} y_n$ and $y_n \in \frac{1}{2^n}(A \cap B_0(\epsilon))$.

(d) Argue that $\frac{1}{2} B_0(\epsilon) \subseteq A$ and hence $0 \in \mathring{A}$.

(e) If $x \in \mathring{\bar{A}}$, show that $x \in \mathring{A}$ [Hint: translate so that x becomes 0].

2) Suppose X is a vector space that is complete under two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. If $\exists\, c < \infty$ s.t. $\|x\|_2 \leq c\|x\|_1$ $\forall x \in X$, show that $\exists\, c' < \infty$ s.t. $\|x\|_1 \leq c'\|x\|_2$.

3) Suppose $x = (x_1, x_2, \ldots)$ with $x_k \in \mathbb{R}$ is such that $\sum_{n=1}^{\infty} x_n y_n$ converges for any $y = (y_1, y_2, \ldots)$ satisfying $\sum y_n^2 < \infty$. Show that $\sum x_n^2 < \infty$.

~~4) Let X, Y be Banach spaces and let T: X → Y be a bounded linear transformation. Assume that T is injective and let W = T(X).~~

4) Let $T: X \to Y$ be a one-one, onto bounded linear transformation (X, Y are Banach spaces). Show that $T^{-1}: Y \to X$ is a bounded linear transformation.

5) Suppose $T: X \to Y$ is a one-one, bounded linear transformation. (X, Y - Banach spaces) Let $W = T(X)$. Can we say that $T^{-1}: W \to X$ is a bounded linear transformation?

Functional analysis: Problem set 3

1) (a) If A is convex ($A \subseteq X$, a vector space), then $\mathring{A}$ is convex

(b) Give example of a convex set A such that $\mathring{A} = \phi$

2) X - a Banach space (over $\mathbb{R}$). $L : X \to \mathbb{R}$ is linear and $Lx \leq C$ $\forall x \in B_{x_0}(r)$ for some $x_0 \in X$ some $r > 0$, some

Show that $L \in X^*$

3) X - a Banach space. $\mu : X \to \mathbb{R}$ is a perfectly convex function such that ~~$\mu(\lambda x) = |\lambda| \mu$~~ $\mu(\lambda x) = \lambda \mu(x)$ for all $x \in X$ and $\forall \lambda \geq 0$. Show that μ is continuous.

4) X - a Banach space over $\mathbb{R}$.

(a) Given $n \geq 1$, $x_1, \dots, x_n \in X$ linearly independent, $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, $\exists$ $L \in X^*$ such that $L(x_k) = \alpha_k$, $1 \leq k \leq n$.

(b) Is it true for infinitely many $x_1, x_2, \dots \in X$ and $\alpha_1, \alpha_2, \dots \in \mathbb{R}$.

5) Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that f is convex, g is concave and $g \leq f$. Show that there exist $m, c \in \mathbb{R}$ such that $g(x) \leq mx + c \leq f(x)$ for all $x \in \mathbb{R}$.

6)* Recall the Banach space $\ell^\infty = \{ \underline{x} = (x_1, x_2, \dots) \mid x_k \in \mathbb{R}, \sup_{k \leq \infty} |x_k| < \infty \}$ with $\|\underline{x}\|_\infty = \sup_i |x_i|$

Show that there exists $L \in (\ell^\infty)^*$ such that

(a) $L((x_1, x_2, \dots)) = L(x_2, x_3, \dots)$ $\forall \underline{x} = (x_1, x_2, \dots)$.

(b) $\liminf_{n \to \infty} x_n \leq L(\underline{x}) \leq \limsup_{n \to \infty} x_n$ $\forall \underline{x} \in \ell^\infty$