REPRESENTATION THEORY ELECTIVE COURSE

MID-TERM EXAM

INSTITUTE OF MATHEMATICAL SCIENCES, AUGUST–NOVEMBER 2009

24 SEPTEMBER 2009, 1530 TO 1745 HRS, MATSCIENCE ROOM 123

Please hand in your paper no later than at 1745. Answer in the space provided. Sheets for rough work are provided separately and should not be handed in.

(1) Prove or disprove: an Artinian subring of a division ring is a division ring.

Solution: The statement is true.
Proof: Let A be an Artinian subring (containing the identity) of a division ring. Let a be in A. Multiplication by a (on, say, the left) is an injective endomorphism of A. Since A is Artinian this is also surjective. Which means that 1 is in the image and a^{-1} belongs to A. □

(2) Let p be a prime and G be a p-group not admitting $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ as a quotient. What can you say about G?

SOLUTION: We claim that G is cyclic.
PROOF: Proceed by induction on the order of G. If G is the trivial group, then of course it is cyclic, which proves the base case of the induction. Now to the induction step.

The centre of a p-group being non-trivial, there exists a non-trivial central element of order p, say g, in G. The hypothesis passes to $G/\langle g \rangle$, so by induction $G/\langle g \rangle$ is cyclic. If $a \in G$ is such that its image in $G/\langle g \rangle$ is the generator, we have $G = \langle a, g \rangle$ and G is abelian.

If $G = \langle a \rangle$, then of course G is cyclic and we are done. Otherwise $g \notin \langle a \rangle$, and $G/\langle a^p \rangle \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. $\square$

(3) Let V be a finite dimensional vector space over a division ring D. Let A be a subring of D-endomorphisms of V. Assume that A is 2-*transitive*, i.e., given any two linearly independent elements v, w of V and any two elements v', w' of V, there exists a in A such that $av = v'$ and $aw = w'$.[1] Compute the commutant and bicommutant of A, even A itself.

Solution:

Claim 1: V is a simple A-module. (This needs only 1-transitivity.)
Proof: Let W be an A-submodule, $0 \neq W \subseteq V$. Given $v \in V$ and $0 \neq w \in W$, there exists, by 1-transitivity, $a \in A$ such that $aw = v$, but then aw belongs to W. Thus $W = V$. $\square$ (Claim 1)

Now, by Schur, $\operatorname{End}_A V$ is a division ring, say E. Since $A \subseteq \operatorname{End}_D V$, it is clear that $E \supseteq D$.
Claim 2: $E = D$. (In other words, the commutant of A is D.)
Proof: Let $e \in E$.

If, for some $v \in V$, the elements ev and v are D-linearly independent, then we have a contradiction: by 2-transitivity, there exists a in A such that $av = 0$ but $a(ev) \neq 0$, but then $a(ev) = e(av) = e0 = 0$, a contradiction.

Let $0 \neq v$ be in V. Let $d \in D$ be such that $ev = dv$. Now $e - d$ is an A-endomorphism of V and has non-trivial kernel. Since the kernel is an A-submodule, it is all of V (by Claim 1), and so $e = d$. $\square$ (Claim 2)

Conclusion: the commutant of A is D (Claim 2), so the bicommutant is $\operatorname{End}_D V$; finally, by the density theorem and the hypothesis of finite dimensionality of V (and Claim 1), it follows that $A = \operatorname{End}_D V$. $\square$

[1] This should be taken to mean that A is also 1-transitive (to cover for the situation when there may not exist two linearly independent elements, lest the hypothesis on A become vacuous).

(4) Let A be an algebra of finite dimension over an algebraically closed field k. Let V_1, …, V_m be simple A-modules no two of which are isomorphic. Consider the commutant C of the ring of homotheties of the module $V_1 \oplus \cdots \oplus V_m$. List all the simple C-modules and their dimensions (over k).

SOLUTION: By the universal properties of direct sum and direct product:

$$\operatorname{End}_A(V_1 \oplus \cdots \oplus V_m) = \prod_{1 \leq i \leq m} \prod_{1 \leq j \leq m} \operatorname{Hom}_A(V_i, V_j)$$

By Schur: $\operatorname{Hom}_A(V_i, V_j)$ is zero if $i \neq j$; moreover, in the case $i = j$, $\operatorname{Hom}_A(V_i, V_i) = \operatorname{End}_A V_i$ is a division ring.

Since A is a k-algebra, we have $k \subseteq \operatorname{End}_A V_i \subseteq \operatorname{End}_k V_i$, so the division ring $\operatorname{End}_A V_i$ contains k in its centre. Since each V_i is of finite dimension over k, it follows that $\operatorname{End}_A V_i$ is of finite dimension as a k-vector space. Since k is algebraically closed, we conclude that $\operatorname{End}_A V_i = k$ for all i.

We have thus proved that the commutant $C := \operatorname{End}_A(V_1 \oplus \cdots \oplus V_m)$ is $\prod_{1 \leq i \leq m} \operatorname{End}_A V_i = \prod_{1 \leq i \leq m} k$. In particular, C is commutative. Let us write k_i for $\operatorname{End}_A V_i$. Every ideal of C is of the form $\oplus_{i \in S} k_i$, where S is a subset of $\{1, \ldots, m\}$. Thus the maximal ideals are precisely $\oplus_{i \neq i_0} k_i$, as i_0 varies over 1, …, m.

We conclude that there are precisely m different simple C-modules, namely $k_{i_0} \simeq C / \oplus_{i \neq i_0} k_i$, each of dimension 1 over k. (The modules are non-isomorphic because their annihilators are different. The annihilators are the respective maximal ideals themselves, C being commutative.) $\square$

(5) Prove or disprove: a module is semisimple if its opposite is so.

Solution: The statement is false.
Counter-example: Let V be the space of 2×1 matrices over a field k. Let A be the k-algebra of 2×2 upper triangular matrices over k. Then V is an A-module by left multiplication. It is not semi-simple: the only simple submodule is the space of matrices with vanishing $(2, 1)$ entries.

We claim that the commutant of A is just k (i.e., scalar matrices). The commutant being a k-subalgebra of $\operatorname{End}_k V$, this is just a calculation with matrices: on the one hand, A contains the diagonal matrices, so the commutant is contained in the space of diagonal matrices; on the other hand, we have

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

which proves the claim.

Thus the opposite of V is a vector space over a field. In particular, it is semisimple. $\square$