3. Solvable, super-solvable, and nilpotent groups

Let G be a group.

3.1. **Commutators; solvability and nilpotency.** The *commutator* (g, h) of elements g, h of G is defined to be the element $ghg^{-1}h^{-1}$. If H and K are subgroups, the subgroup generated by all commutators (h, k) with $h \in H$ and $k \in K$, denoted (H, K), is the *commutator group* of H and K. It is normal, respectively characteristic,[4] if H and K are so.

The subgroup (G, G) is called the *commutator subgroup* or *derived subgroup*. It is characteristic and the quotient by it is abelian. In fact, any abelian quotient of G factors through $G/(G, G)$. Taking successive commutators, we can generate two descending series of characteristic subgroups of G:

- $\mathfrak{D}^0G := G$, $\mathfrak{D}^1G := (G, G)$, ..., $\mathfrak{D}^{i+1}G := (\mathfrak{D}^iG, \mathfrak{D}^iG)$, ...
- $\mathfrak{C}^0G := G$, $\mathfrak{C}^1G := (G, G)$, ..., $\mathfrak{C}^{i+1}G := (G, \mathfrak{C}^iG)$, ...

We call G *solvable* if $\mathfrak{D}^nG = \{1\}$ for some n; *nilpotent* if $\mathfrak{C}^nG = \{1\}$ for some n. Solvability and nilpotency

3.2. **Characterizing solvability.** In order to formulate useful characterizations of solvability, consider the existence of a series of subgroups

$$\{1\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots \subseteq G_n = G \tag{3.1}$$

where each G_i is normal in G_{i+1} and the quotients G_{i+1}/G_i are abelian. Consider also the existence of such series satisfying stronger conditions:

$$\text{the } G_i \text{ are normal in } G \tag{3.2}$$

$$\text{the quotients } G_{i+1}/G_i \text{ are cyclic} \tag{3.3}$$

Let G be solvable and choose n such that $\mathfrak{D}^nG = \{1\}$. Then the series consisting of the subgroups $G_i := \mathfrak{D}^{n-i}G$ is of the form (3.1); in fact, it evidently satisfies the stronger condition (3.2). On the other hand, if a series like (3.1) exists, then, firstly, since $(G_{i+1}, G_{i+1}) \subseteq G_i$, it follows that $\mathfrak{D}^nG = \{1\}$, so G is solvable; next, we can refine the given series by inserting between G_i and G_{i+1} subgroups normal in G_{i+1} in such a way that successive quotients in the refined series are cyclic. The refined series thus satisfies the stronger condition (3.3).

We've thus proved the equivalence of the following conditions on a group G:

- G is solvable
- there exists a subgroup series as in (3.1)
- there exists a subgroup series as in (3.1) satisfying (3.2)
- there exists a subgroup series as in (3.1) satisfying (3.3)

Solvablility does not however imply the existence of a subgroup series (3.1) satisfying simultanenously both (3.2) and (3.3). If such a series exists, then we call G *super-solvable*. It is routine to verify the following: Super-solvability

- Super-solvable groups are of course solvable.
- Subgroups and quotients of solvable (respetively super-solvable) groups are solvable (respectively super-solvable).
- Extensions of solvable groups by solvable groups are solvable.
- Cyclic extensions of super-solvable groups are super-solvable.
- A solvable (respectively super-solvable) group can be constructed by successive abelian (respetively cyclic) extensions starting from the trivial group.

[4]A subgroup is *characteristic* if it is invariant under all automorphisms of the group.

3.3. **An equivalent condition for nilpotency.** Consider the following ascending series of subgroups (where $\mathfrak{z}(G)$ denotes the centre of G):

(3.4) $\quad G_0 := \{1\}, \quad G_1 := \mathfrak{z}(G), \quad G_{i+1}$ is defined by $G_{i+1}/G_i = \mathfrak{z}(G/G_i)$.

It is routine to verify the following:

- G is nilpotent if and only if $G_n = G$ for some n in the series (3.4). In fact, $\mathfrak{C}^n G = \{1\}$ if and only if $G_n = G$.
- Subgroups and quotients of nilpotent groups are nilpotent.
- Central extensions of nilpotent groups are nilpotent.
- Nilpotent groups are super-solvable (and so solvable).
- p-groups are nilpotent by (1.8).

3.4. **Structure of finite nilpotent groups.**

Theorem 3.1. *A finite nilpotent group has a unique Sylow p-subgroup for every prime p. In particular, it is a direct product of its Sylow p-subgroups. Conversely, any product of p-groups is nilpotent.*

Proof. Let P be a Sylow p-subgroup and N its normalizer. On the one hand, by the last item of (1.10), N is its own normalizer; on the other hand, by the lemma below, N is strictly contained in its normalizer if N is proper; which leads us to conlcude that N is the whole group, or, in other words, that P is normal. Since Sylow p-subgroups are all conjugate (1.10), it follows that P is the unique Sylow p-subgroup. The second assertion now follows from an elementary calculation: see 1.4.11. $\square$

Lemma 3.2. *The normalizer in a nilpotent group of a proper subgroup H strictly contains H.*

Proof. Let G_1 be the centre of G, and G_2 the subgroup of G such that G_2/G_1 is the center of G/G_1, ... Then G_n equals G for some n—this is what it means for G to be nilpotent. Since H is proper, there is an i such that $G_i \subseteq H$ and $G_{i+1} \not\subseteq H$. Choose n in $G_{i+1} \setminus H$. Then, for h in H, $(n, h) = nhn^{-1}h^{-1}$ belongs to G_i and so to H, which means that nhn^{-1} belongs to H. Thus n normalizes H. $\square$

3.5. **Exercises.**

3.5.1. The dihedral group D_n defined in §2.1.1 is super-solvable. It is nilpotent if and only if n is a power of 2.

3.5.2. A maximal proper subgroup of a nilpotent group is normal. In particular, a subgroup whose index is prime is normal (in a nilpotent group).

3.5.3. Let G be a p-group. Then:

- Every subgroup of index p is normal.
- If G is not cyclic, then it has $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ as a quotient.
- If $G/(G, G)$ is cyclic, then G is cyclic.