[s:rad]

## 10. The Radical (Après Bourbaki Algebra Chapter 8 §6)

Let $A$ be a ring (with identity according to our convention) and $M$ an $A$-module. For additive subgroups $U$ and $V$ of $A$ and $M$ respectively, we denote by $UV$ the subset of $M$ consisting of finite sums $\sum_i u_i v_i$ with $u_i$ and $v_i$ in $U$ and $V$ respectively. Thus $UV$ is a submodule if $U$ is a left ideal; the product of left ideals is a left ideal; the product of a left ideal and a right ideal is a two-sided ideal.

[ss:nil]

10.1. **Nil and nilpotent ideals.** An element $a$ (respectively, an ideal $\mathfrak{a}$) of $A$ is *nilpotent* if $a^n = 0$ (respectively, $\mathfrak{a}^n = 0$) for some $n \geq 1$. An ideal consisting of nilpotent elements is a *nil ideal*. We have:

- If $a$ is nilpotent, then $1 - a$ is a unit with inverse $1 + a^1 + a^2 + \ldots$ (note that the sum is finite).
- An ideal is nilpotent if and only if there exists an $n$ such that the product of any $n$ elements all belonging to the ideal vanishes.
- A nilpotent ideal is clearly nil. But not every nil ideal is nilpotent. In fact, in §**??** we exhibit a non-zero nil ideal $\mathfrak{a}$ such that $\mathfrak{a}^2 = \mathfrak{a}$.

[ss:radm]

10.2. **The radical of a module.** The *radical* $\mathfrak{Rad}\, M$ of a module $M$ is the intersection of all its maximal submodules, or, equivalently, the intersection of kernels of all homomorphisms into simple modules. We have:

- $\mathfrak{Rad}\, M$ vanishes if and only if $M$ is the submodule of a direct product of simple modules; in particular, a semisimple module has trivial radical.
- Homomorphisms map radicals into radicals. If a submodule $N$ is contained in $\mathfrak{Rad}\, M$, then $\mathfrak{Rad}(M/N) = (\mathfrak{Rad}\, M)/N$. The radical is the smallest submodule $N$ such that $\mathfrak{Rad}(M/N)$ vanishes. (however, just because a submodule $N$ contains $\mathfrak{Rad}\, M$, it does not mean that $\mathfrak{Rad}(M/N)$ vanishes.)
- $\oplus \mathfrak{Rad}\, M_i = \mathfrak{Rad} \oplus M_i \subseteq \mathfrak{Rad} \prod M_i \subseteq \prod \mathfrak{Rad}\, M_i$.
- Let $M$ be of finite type. Then
  - $\mathfrak{Rad}\, M = M$ implies $M = 0$; more generally, $N + \mathfrak{Rad}\, M = M$ for a submodule $N$ implies $N = M$. (If $N \subsetneq M$, then choose $P$ maximal submodule with $P \supseteq N$—this uses the finite generation of $M$; then $\mathfrak{Rad}\, M \subseteq P$, so $N + \mathfrak{Rad}\, M \subseteq P$.)
  - $x$ in $M$ belongs to $\mathfrak{Rad}\, M$ if and only if for any finite set $x_1, \ldots, x_n$ of generators of $M$ and any set $a_1, \ldots, a_n$ of elements of $A$, the set $x_1 + a_1 x, \ldots, x_n + a_n x$ is also a set of generators. (If $x_1, \ldots, x_n$ are generators and $a_1, \ldots, a_n$ are such that $x_1 + a_1 x, \ldots, x_n + a_n x$ are not, then choose $N$ maximal submodule containing $x_1 + a_1 x, \ldots, x_n + a_n x$. Then $x \notin N$, for otherwise $x_1, \ldots, x_n$ belong to $N$, a contradiction. Conversely, suppose $x$ is not in the radical. Then choose maximal $N$ such that $x \notin N$. Let $a_1$ be such that $x_1 + N = a_1 x + N$ (such an $a_1$ exists since $M/N$ is simple and $x \notin N$). Let $a_2$, $\ldots$, $a_n$ be choosen analogously with respect to $x_2, \ldots, x_n$. Then $x_1 - a_1 x, \ldots, x_n - a_n x$ all belong to $N$ and therefore do not generate $M$.)

[ss:rada]

10.3. **The radical of a ring.** The *radical* $\mathfrak{Rad}\, A$ of $A$ is defined to be its radical as a left module over itself: $\mathfrak{Rad}\, A := \mathfrak{Rad}\,_A A$. The annihilator of a simple module (in other words, a primitive ideal) is evidently the intersection of the annihilators of the non-zero elements of the module; these being all maximal left ideals, we get

$$\mathfrak{Rad}\, A = \text{intersection of annihilators of simple (respectively, semisimple) modules}$$

We have:

- $\mathfrak{Rad}\, A$ is a two sided ideal. For, annihilators of modules are two sided ideals.
- $(\mathfrak{Rad}\, A)M \subseteq \mathfrak{Rad}\, M$, for $M/\mathfrak{Rad}\, M$ is a submodule of a direct product of simple modules and so killed by $\mathfrak{Rad}\, A$. Equality need not hold even in very good cases: e.g. $A = \mathbb{Z}$ and $M = \mathbb{Z}/p^n\mathbb{Z}$ with $n \geq 1$.
- NAKAYAMA'S LEMMA: If $M$ is a of finite type and $N$ a submodule such that $N + (\mathfrak{Rad}\, A)M = M$, then $N = M$. (For, $(\mathfrak{Rad}\, A)M \subseteq \mathfrak{Rad}\, M$. See the relevant sub-item of the last section.)
  - Let $M$ be of finite type and $\mathfrak{m}$ a right ideal contained in the radical $\mathfrak{Rad}\, A$. If $A_A/\mathfrak{m} \otimes_A M = 0$, then $M = 0$. (For, $0 = A_A/\mathfrak{m} \otimes_A M \simeq M/\mathfrak{m}M$ menas $(\mathfrak{Rad}\, A)M = M$.)
  - Let $u : M \to N$ be a $A$-linear map of modules. Let $\mathfrak{m}$ be a right ideal contained in $\mathfrak{Rad}\, A$, $N$ be finitely generated, and $\mathrm{id} \otimes u : A_A \otimes_A M \to A_A \otimes_A N$ be surjective. Then $u$ is surjective.
- Let $\mathfrak{a}$ be a two sided ideal of $A$. Then $\mathfrak{Rad}(A/\mathfrak{a}) \supseteq (\mathfrak{Rad}\, A + \mathfrak{a})/\mathfrak{a}$. If $\mathfrak{a} \subseteq \mathfrak{Rad}\, A$, then $\mathfrak{Rad}(A/\mathfrak{a}) = \mathfrak{Rad}\, A/\mathfrak{a}$. (The $A$-module structure of $A/\mathfrak{a}$ coincides with that of its structure as a module over itself. Now use the relevant items from the last section.)
- The $\mathfrak{Rad}\, A$ is the smallest two sided ideal such that $A/\mathfrak{Rad}\, A$ has no radical. (By the previous item it follows that $A/\mathfrak{Rad}\, A$ has no radical as a ring. Conversely, if $\mathfrak{Rad}\,(A/\mathfrak{a}) = 0$, then $(\mathfrak{Rad}\, A + \mathfrak{a})/\mathfrak{a} = 0$ (previous item), so $\mathfrak{Rad}\, A \subseteq \mathfrak{a}$.)

**Theorem 10.1.** *An element $x$ of the ring $A$ belongs to the radical if and only if $1 - ax$ has a left inverse for every $a$ in $A$.* [t:jacperlis]

*Proof.* This follows from the characterization in the last section of elements belonging to the radical of a module of finite type. □

We have as corollaries:

- $\mathfrak{Rad}\, A$ is the largest left ideal $\mathfrak{a}$ such that $1 - x$ has a left inverse for every $x$ in $\mathfrak{a}$.
- $\mathfrak{Rad}\, A$ is the largest two sided ideal $\mathfrak{a}$ such that $1 - x$ is invertible for every $x$ in $\mathfrak{a}$. (By the theorem, it suffices to show that $1 - x$ is invertible when $x$ is in $\mathfrak{Rad}\, A$. We know that it has a left inverse, say $y$: $y(1 - x) = y - yx = 1$. We will show that $y$ is invertible, i.e., it also has a left inverse. It will then follow that $(1 - x) = y^{-1}$ is also invertible. Since $z := 1 - y = -yx$ belongs to $\mathfrak{Rad}\, A$, there exists $y'$ such that $y'$ is a left inverse for $1 - z = y$.)
- $\mathfrak{Rad}\,(A^{\mathrm{opp}}) = \mathfrak{Rad}\, A$. (This is a consequence of the previous item.)
- Any nil ideal (left, right, or two sided) is contained in the radical. (The previous item is used in the proof that a right nil ideal is contained in the radical.)
- The radical of a direct product of rings is the direct product of the radicals.

Not every nilpotent element is contained in the radical (e.g., in $M_n(\mathbb{C})$). But a nilpotent central element belongs to the radical, for the ideal it generates is nil. $\mathfrak{Rad}\, A$ is not necessarily a nil ideal; in particular, not necessarily nilpotent. It can happen that $\mathfrak{Rad}\, A^2 = \mathfrak{Rad}\, A$ even if $\mathfrak{Rad}\, A$ is a nil ideal.

**Theorem 10.2.** *A left ideal $\mathfrak{l}$ is contained in $\mathfrak{Rad}\, A$ if and only if for every finitely generated non-zero module $M$ we have $\mathfrak{l}M \neq M$*

*Proof.* The 'only if' part is Nakayama's lemma. For the if part, the hypothesis implies that $\mathfrak{l}N = 0$ for every simple module $N$ (because simple modules are cyclic and contain no non-trivial proper submodules), which means $\mathfrak{l} \subseteq \mathfrak{Rad}\, A$. □

As examples, we have:

- Let $A$ be the ring $k[[X_1, \ldots, X_n]]$ of formal power series in finitely many variables over a field $k$. The units in $A$ are the series with non-zero constant term. The elements with vanishing constant term constitute the unique maximal ideal of $A$, which therefore is the radical. There are no non-trivial nilpotent elements in $A$.

  The quotient field of $A$ has of course no radical. Thus the sub-ring of a ring without radical could well have radical.
- Let $C$ be an integral domain and $B$ the polynomial ring $C[X_1, \ldots, X_n]$ in finitely many variables over $C$. Then, if $n > 0$, $B$ has trivial radical: in fact, for $0 \neq f$, we have $\deg(1 - fg) > 0$ and so $1 - fg$ is not a unit for $g$ any element of positive degree.

  Let $k$ be a field. Then $k[X_1, \ldots, X_n]$ is without radical. But its over ring $k[[X_1, \ldots, X_n]]$ has non-trivial radical intersecting $k[X_1, \ldots, X_n]$ non-trivially.
- Let $k$ be a field, $S$ a set, and $A$ the ring of $k$ valued functions on $S$. Then $A$ is without radical. Indeed, the evaluation at any point $s$ of $S$ gives a morphism $A \to k$, whose kernel is therefore a maximal ideal. The intersections of these maximal ideals as $s$ varies over $S$ is clearly 0.

[p:radpid]

**Proposition 10.3.** *Let $A$ be a principal ring.*

(1) *$A$ is without radical if and only if either $A$ is a field or $A$ has infinitely many maximal ideals.*
(2) *$A/Ax$ is without radical if and only if $x$ is square free.*

*Proof.* Let $(p_\alpha)$ be a system of representatives of maximal elements. The maximal ideals of $A$ are $Ap_\alpha$. In order that □

[ss:radart]

10.4. **The radicals of Artinian rings and modules.**

[t:artrad]

**Theorem 10.4.** *Let $A$ be Artinian. Then $\mathfrak{Rad}\, A$ is the largest two sided nilpotent ideal of $A$.*

*Proof.* Any nil ideal (one-sided or two-sided) is contained in the radical, as has already been observed in the last subsection. It suffices to prove therefore that $\mathfrak{Rad}\, A$ is nilpotent (we have also observed that $\mathfrak{Rad}\, A$ is a two-sided ideal, being the annihilitor of all simple modules). Set $\mathfrak{r} := \mathfrak{Rad}\, A$. Choose $n$ large enough so that $\mathfrak{r}^n = \mathfrak{r}^{n+1} = \ldots =: \mathfrak{a}$. It suffices to assume that $\mathfrak{a} \neq 0$ and arrive at a contradiction.

Assume $\mathfrak{a} \neq 0$. Choose a minimal left ideal $\mathfrak{l}$ with the property that $\mathfrak{a}\mathfrak{l} \neq 0$ (such an ideal exists by the Artinian hypotheis: observe that $\mathfrak{a}A = \mathfrak{a} \neq 0$, so the collection of ideals with the property is non-empty). Now, on the one hand, $\mathfrak{a}(\mathfrak{r}\mathfrak{l}) = (\mathfrak{a}\mathfrak{r})\mathfrak{l} = \mathfrak{a}\mathfrak{l} \neq 0$, so that $\mathfrak{r}\mathfrak{l}$ has the property; on the other, $\mathfrak{r}\mathfrak{l} \subseteq \mathfrak{l}$. So $\mathfrak{r}\mathfrak{l} = \mathfrak{l}$ by the minimality of $\mathfrak{l}$.

We claim now that $\mathfrak{l}$ is finitely generated. It will then follow, by Nakayama's lemma, that $\mathfrak{l} = 0$, which is a contradiction, since $\mathfrak{a}\mathfrak{l} \neq 0$ by choice of $\mathfrak{l}$, and the proof will be over.

To prove the claim, we prove in fact that $\mathfrak{l}$ is cyclic. Indeed, there exists $x \in \mathfrak{l}$ such that $\mathfrak{a}x \neq 0$ (by the choice of $\mathfrak{l}$); now $Ax$ is such that $\mathfrak{a}Ax \neq 0$ and $Ax \subseteq \mathfrak{l}$, so that $Ax = \mathfrak{l}$ by the minimality of $\mathfrak{l}$. □

[c:commartrad]

**Corollary 10.5.** *The radical of a commutative Artinian ring equals the subset of its nilpotent elements.*

*Proof.* By Artinianness, the radical is nilpotent. By commutativity, the ideal generated by a nilpotent element is nilpotent, and so contained in the radical. □

[t:radartmod]

**Theorem 10.6.** *$M$ is semisimple of finite length if and only if it is Artinian and $\mathfrak{Rad}\, M = 0$.*

*Proof.* A finite length module is Artinian (and Noetherian); the radical of a semisimple module vanishes. Conversely, suppose that $M$ is Artinian and $\mathfrak{Rad}\, M = 0$. Consider, using Artinianness, a smallest element—call it $N$—of the set of submodules that are written as finite intersections of maximal submodules. (This collection is non-empty, $M$ itself being the intersection of the empty collection.) If $N \neq 0$, choose $0 \neq n \in N$. Since $\mathfrak{Rad}\, M = 0$, there exists a maximal submodule $K$ such that $n \notin K$. Now, adding $K$ to the collection from which we obtained $N$, we get a contradiction to the minimality of $N$, since $K \cap N \subsetneq N$. This shows $N = 0$.

In other words, we have shown that there exist finitely many maximal submodules $N_1, \ldots, N_k$ of $M$ such that their intersection is 0. This means $M \hookrightarrow M/N_1 \oplus \cdots \oplus M/N_k$. So $M$ is of finite length and semisimple (since so is $M/N_1 \oplus \cdots \oplus M/N_k$). □

We have, as corollories:

- If $M$ is Artinian, then $M/\mathfrak{Rad}\, M$ is semisimple of finite length.
- $A$ is semisimple if and only if it is Artinian with trivial radical.
- If $A$ is Artinian, $A/\mathfrak{Rad}\, A$ is semisimple.
- $A$ is simple if and only if it is Artinian and its only two sided ideals are 0 and itself.
- The following are equivalent for a commutative ring:
  - it is Artinian and contains no non-trivial nilpotent elements;
  - it is semisimple;
  - it is a finite direct product of fields.
- Let $k$ be a field and $A$ a commutative finite dimensional $k$-algebra. Assume that $\mathfrak{Rad}\, A = 0$. Then $A$ is a finite direct product of fields, each of which is a finite extension of $k$.

[ss:modoverart]

10.5. **Modules over Artinian rings.**

**Proposition 10.7.** *Let $A$ be an Artinian ring and $M$ an $A$-module. Then the following are equivalent:*

- *$M$ is semisimple.*
- *$(\mathfrak{Rad}\, A)M = 0$.*
- *$A_M$ is semisimple.*

*Proof.* If $M$ is semisimple, then $\mathfrak{Rad}\, M = 0$; in general, $(\mathfrak{Rad}\, A)M \subseteq \mathfrak{Rad}\, M$, so the first implies the second. If $A_M$ is semisimple then of course $M$ is so (being a module for $A_M$).

The hypotheis that $A$ is Artinian will be used only now. Let $(\mathfrak{Rad}\, A)M = 0$. Then $A_M$ is a quotient of $A/\mathfrak{Rad}\, A$. But $A/\mathfrak{Rad}\, A$ is semisimple, it being Artinian and without radical. Hence so is $A_M$. □

[p:artsimplemod]

**Proposition 10.8.** *Over an Artinian ring, there exist only finitely many isomorphism classes of simple modules, this number being equal to the number of simple components of $A/\mathfrak{Rad}\, A$.*

*Proof.* Any simple module is a module also for $A/\mathfrak{Rad}\, A$. And $A/\mathfrak{Rad}\, A$ is a semisimple ring. □

[p:abynss]

**Proposition 10.9.** *Let $A$ be a ring admitting a two sided nilpotent ideal $\mathfrak{n}$ such that $A/\mathfrak{n}$ is semisimple (e.g., an Artinian ring). For any $A$-module, the following conditions are equivalent:*

- *$M$ is of finite length*
- *$M$ is Artinian*
- *$M$ is Noetherian*

*Proof.* If $M$ is of finite length then of course it is both Artinian and Noetherian. Now suppose that $\mathfrak{n}^p = 0$ and that $M$ is Artinian (respectively, Noetherian). Consider the filtration $M \supseteq \mathfrak{n}M \supseteq \mathfrak{n}^2 M \supseteq \ldots \supseteq \mathfrak{n}^{p-1}M \supseteq \mathfrak{n}^p M = 0$. The quotients are $M/\mathfrak{n}M$, $\mathfrak{n}M/\mathfrak{n}^2M$, …, $\mathfrak{n}^{p-1}M/\mathfrak{n}^pM$. These being modules over the semisimple ring $A/\mathfrak{n}$, they are on the one hand semisimple. On the other, being sub-quotients of $M$, they are Artinian (respectively, Noetherian). But a semisimple module is of finite length if it is Artinian (or Noetherian). So each of the quotients is of finite length and therefore so is $M$. □

**Corollary 10.10.** *A finitely generated module over an Artinian ring is of finite length. In particular, the ring itself is of finite length. Artinian rings are therefore Noetherian.*

*Proof.* A finitely generated module is Artinian. Now apply the proposition. □

10.6. **Exercises.**

10.6.1. Let $A$ be a ring.

- Let $Z$ be the centre of $A$. Show that $Z \cap \mathfrak{Rad}\, A$ is contained in $\mathfrak{Rad}\, Z$.
- Show that $\mathfrak{Rad}\, A$ does not contain any non-zero idempotent.

10.6.2. If $\mathfrak{Rad}\, M = 0$, then $\mathfrak{Rad}\, A_M = 0$.

10.6.3. Let $A$ be a ring such that $A/\mathfrak{Rad}\, A$ is semisimple. Then for any $A$-module $M$, we have $\mathfrak{Rad}\, M = (\mathfrak{Rad}\, A)M$. Hint: Observe that $M/(\mathfrak{Rad}\, A)M$ is a $A/\mathfrak{Rad}\, A$-module.