8. The commutant and bicommutant of a semisimple module (Après Bourbaki Algebra Chapter 8 §4)

[s:commbicomm]

Throughout this section A denotes a ring, M an A-module, A_M the ring of homotheties of M (namely the image in $\mathrm{End}_{\mathbb{Z}} M$ of A), C the commutant in $\mathrm{End}_{\mathbb{Z}} M$ of A, and B the commutant in $\mathrm{End}_{\mathbb{Z}} M$ of C. We call B the *bicommutant of M*. Evidently $A_M \subseteq B$.

[ss:bssmod]

8.1. The bicommutant of a semisimple module.

Let M be semisimple and $M = \oplus_\lambda M_\lambda$ its direct sum decomposition into isotypic components.

- M is simple (respectively, isotypic) for B if and only if it is so for A.
- The product $B_M \to \prod_\lambda B_{M_\lambda}$ of the natural maps $B_M \to B_{M_\lambda}$ (induced by the inclusion of M_λ as a direct summand) is an isomorphism.
- If M is isotypic of type S (where S is a simple module), then the map $B_M \to B_S$ (induced by the inclusion of S any which way) is an isomorphism.

[ss:density]

8.2. The density theorem.

[t:density]

Theorem 8.1. (The density theorem) *The ring A_M of homotheties of a semisimple module M is dense in its bicommutant B: given an element of b in B and a finite sequence $x_1, \ldots, x_n$ of elements of M, there exists an a in A such that $bx_1 = ax_1, \ldots, bx_n = ax_n$.*

Proof. The module $M^{\oplus n}$ is also semisimple. Set $x := (x_1, \ldots, x_n)$. The submodule Ax of $M^{\oplus n}$ is a direct summand (by the semisimplicity assumption) and therefore stable under the bicommutant of $M^{\oplus n}$ (Proposition 5.4). The bicommutant of $M^{\oplus n}$ is however the ring of homotheties when it is considered a B-module (Lemma 5.5). Thus $b \in B$ acting diagonally on $M^{\oplus n}$ belongs to the bicommutant. So $bx \in Ax$, which means there exists a such that $bx = ax$, and so $bx_1 = ax_1, \ldots, bx_n = ax_n$. □

We get the following corollaries:

- If M is semisimple and its opposite is finitely generated, then $A_M = B_M$. Indeed, the action of $b \in B_M$ is determined by its action on the finite generators of the opposite, but there exists an element a which matches its action on any finite set of elements.
- Let $S_1, \ldots, S_n$ be simple A-modules, no two being isomorphic. Assume that their opposites are finitely generated. Given $a_1, \ldots, a_n$ elements in the rings $A_{S_1}, \ldots, A_{S_n}$ of homotheties, there exists a in A such that its image in A_{S_i} is a_i for $1 \leq i \leq n$. (Proof: Let $M = S_1 \oplus \cdots \oplus S_n$. The opposite of M is finitely generated since each S_i has that property and any endomorphism of S_i can be lifted to M. By the previous item, $B_M = A_M$. By item (2) in §8.1, $B_M = B_{S_1} \times \cdots \times B_{S_n}$. Since $A_{S_j} \subseteq B_{S_j}$, the result follows.)

[ss:csmod]

8.3. The commutant of a simple module.

[l:schur]
Schur's lemma

Lemma 8.2. (Schur's lemma) *The commutant of the ring of homotheties of a simple module is a division ring.*

The result is easy to prove (and so we omit the proof) but its importance cannot be overestimated. The following simple observation is useful in combination with Schur's lemma (when we have an algebraically closed base field):

> Suppose an algebraically closed field is contained in the centre of a division ring of finite dimension over it. Then the field equals the division ring.

[c:abelian]

Corollary 8.3. *Let A be a commutative algebra over an algebraically closed field k of finite dimension (e.g., the group ring over k of a finite abelian group). Any simple A-module is one dimensional as a k-vector space.*

This holds even when A is just finitely generated as a k-algebra (a version of the Hilbert Nullstellensatz).

Proof. Let M be a simple module. Observe that it must be finite dimensional as a k-vector space: choosing $0 \neq x \in M$ we have $Ax = M$, and so M is k-linearly spanned by $a_1 x, \ldots, a_n x$, where $a_1, \ldots, a_n$ form a k-basis for A. Now, using Schur lemma and the observation above, we conclude that the commutant of A_M is k_M. But A being commutative, this forces $A_M \subseteq k_M$, which means that every k-subspace of M is also an A-submodule. □

[sss:schurdense]

8.3.1. Schur's lemma and the density theorem. We now draw some conclusions by combining Schur's lemma with the density theorem.

[c:l:schur:1]

Corollary 8.4. *Let A be an associative k-algebra with identity and M a simple A-module. Suppose that k is algebraically closed and that M is finite dimensional as a k-module. Then the ring of homotheties is the full ring of k-endomorphisms of M.*

Proof. Apply in turn Schur's lemma, the observation above, and the density theorem. □

A special case of the above result carries a name:

[t:burnside]

Theorem 8.5. (Burnside) *Let G be a group and M a simple G-module over an algebraically closed field k. Assume that M is finite dimensional over k (this is automatic if G is finite). Then the image of G in $\operatorname{End}_k M$ linearly spans $\operatorname{End}_k M$.*

[c:l:schur:2]

Corollary 8.6. *Let k be an algebraically closed field and A a k-algebra. Let M_1, …, M_n be simple A-modules, no two of which are isomorphic, and all of which are finite dimensional over k. Given $\phi_i \in \operatorname{End}_k M_i$, $1 \leq i \leq n$, there exists $a \in A$ such that the action of a on M_i is ϕ_i.*

Proof. This follows from Corollary 8.4 above and the second of the corollaries listed in §8 of the density theorem. □

[p:srfinite]

Proposition 8.7. *Let M be a semisimple A-module and S a simple A-module. Let D be the commutant of S. Then $\operatorname{Hom}_A(S, M)$ and $\operatorname{Hom}_A(M, S)$ are naturally right and left D-vector spaces respectively. We have:*

- *$[M : S] = \dim_D \operatorname{Hom}_A(S, M)$;*
- *there is a unique isomorphism T of $\operatorname{Hom}_A(M, S)$ with the dual space of $\operatorname{Hom}_A(S, M)$ such that $T(u)(v) = uv$ (note: $uv \in \operatorname{Hom}_A(S, S) =: D$).*

Proof. If N be the S-isotypic component of M, then naturally $\operatorname{Hom}_A(M, S) \simeq \operatorname{Hom}_A(N, S)$, $\operatorname{Hom}_A(S, M) \simeq \operatorname{Hom}_A(S, N)$, and $[M : S] = [N : S]$, so we may assume that M is isotypic. The first item now follows from Theorem **??** (b).

By the same theorem, that the map $T(u)(v) = uv$ is a bijection from $\operatorname{Hom}_A(M, S)$ to $\operatorname{Hom}_D(\operatorname{Hom}_A(S, M), \operatorname{Hom}_A(S, S))$. The latter has a natural left D-space structure (namely, $(\lambda(t))(v) = \lambda(t(v))b$), and so the bijection is a left D-space isomorphism: $T(\lambda u)(v) = (\lambda u)v = \lambda(uv) = \lambda(T(u)v) = (\lambda(T(u))v$, so $T(\lambda u) = \lambda(T(u))$.

Finally, the uniqueness of T is evident. □

[c:p:srfinite]

Corollary 8.8. *With hypothesis as in the proposition above, in order for* $[M : S]$ *to be finite, it is necessary and sufficient that* $\dim_D \operatorname{Hom}_A(M, S)$ *be finite. In case the condition is met, then* $[M : S] = \dim_D \operatorname{Hom}_A(M, S)$.