[s:abcomm]

## 5. Commutation (Après Bourbaki Algebra Chapter 8 §1)

Throughout $A$ denotes a ring with identity, $M$ a (left) $A$-module, and $\mathfrak{L}$ the ring $\operatorname{End}_A M$ of $A$-endomorphisms of $M$. The image of $A$ in $\operatorname{End}_{\mathbb{Z}} M$ is denoted $A_M$ and called *the ring of homotheties*. The ring $\mathfrak{L}$ of $A$-endomorphisms is then the centralizer or *commutant* in $\operatorname{End}_{\mathbb{Z}} M$ of the ring of homotheties. Since $\mathfrak{L}$ is a subring of $\operatorname{End}_{\mathbb{Z}} M$ by definition, $M$ is naturally an $\mathfrak{L}$-module, called the *opposite* of the original $A$-module $M$.

The centralizer, again in $\operatorname{End}_{\mathbb{Z}} M$, of $\mathfrak{L}$ is called the *bicommutant of* $M$ and denoted by $B_M$ (or just $B$ if the context is clear). Since $B$ too is a subring of $\operatorname{End}_{\mathbb{Z}} M$, it follows that $M$ is naturally also a $B$-module. Clearly $A_M$ is a subring of $B_M$.

[ss:proj]

5.1. **Projections.** An element $e$ of $\mathfrak{L}$ is a *projection* if it is an idempotent, i.e., $e^2 = e$. Given a projection $e$, we have

- $M = \operatorname{Ker} e \oplus \operatorname{Im} e$;
- if $m = m_1 + m_2$ with $m_1 \in \operatorname{Ker} e$ and $m_2 \in \operatorname{Im} e$, then $em_2 = m_2$ (so the kernel and image of a projection determine it);
- $1 - e$ is a projection with kernel $\operatorname{Im} e$ and image $\operatorname{Ker} e$.
- the left ideal $\mathfrak{L}e$ consists of all endomorphisms that kill $\operatorname{Ker} e$; the right ideal $e\mathfrak{L}$ consists of all endomorphisms with image in $\operatorname{Im} e$.

Let $N$ be a submodule. A projection with image $N$ is called a *projection on* $N$. A *complement* to $N$ is a subspace $P$ such that $N \oplus P = M$. We call $N$ a *direct summand* if there exists a complement to it. A complement to $N$ determines (and is detemined by) a projection on $N$.

A submodule $N$ is a direct summand if and only if every $A$-homomorphism from $N$ can be extended to $M$: indeed, if there is a projection on $N$, we can extend homomorphisms from $N$ to $M$ by pre-composing with the projection; conversely, the extension to $M$ of the identity map of $N$ defines a projection on $N$.

[sss:orthoidem]

5.1.1. Orthogonal idempotents. A family $(e_i)_i$ of idempotents in a ring is called *orthogonal* if $e_ie_j = 0$ for $i \neq j$. A direct sum decomposition $M = \oplus_i M_i$ determines a family of orthogonal idempotents in $\mathfrak{L}$, namely $(p_i)$ where $p_i$ is the projection on $M_i$ with kernel $\sum_{j\neq i} M_j$. Conversely given a family $(e_i)$ of orthogonal idempotents of $\mathfrak{L}$ with the property that, for every $m$ in $M$, the number is finite of idempotents $e_i$ such that $e_im \neq 0$, we get a direct sum decomposition: $M = \oplus_i e_iM$.

[s:commandbicomm]

5.2. **Commutants and bicommutants.**

[p:commtensor]

**Proposition 5.1.** *Let $A$ and $B$ be algebras over a field $k$. Let $C$ and $D$ be subalgebras of $A$ and $B$ respectively; $C'$ and $B'$ the commutants of $C$ and $D$ respectively in $A$ and $B$. Then $C' \otimes B'$ is the commutant of $C \otimes D$ in $A \otimes B$ (all tensor products are over $k$).*

*Proof.* It is clear that $C' \otimes_k B'$ belongs to the commutant, so we need only show the other containment. By considering subspaces complementary to $C'$ and $D'$, we see that $C' \otimes D'$ is the intersection of $C' \otimes B$ and $A \otimes D'$. Let $z = \sum a_i \otimes b_i$ belong to the commutant of $C \otimes D$. We may assume that the $b_i$ are linearly independent (over $k$). For $c$ in $C$, we have $z(c\otimes 1) - (c \otimes 1)z = 0$, which yields $\sum(a_ic - ca_i) \otimes b_i = 0$. Thus $a_ic - ca_i = 0$ for all $i$ and $c$, so the $a_i$ belong to $C'$. This proves that $z \in C' \otimes B$. Similarly $z \in A \otimes D'$. □

**Corollary 5.2.** *The centre of a tensor product is the tensor product of the centres.*

**Proposition 5.3.** *The commutant of the ring of homotheties of the left (respectively right) regular representation consists of right (respectively, left) multiplications by elements of the ring. Thus the ring of homotheties of the regular representation is its own bicommutant.*

[s:abcomm]

5.3. **Direct summands and Bicommutants.**

[p:bfactor]

**Proposition 5.4.** *The bicommutant $B$ preserves direct factors. The restriction $b_N$ of the action of an element $b$ in the bicommutant to a direct summand $N$ maps into the bicommutant of $N$. An $A$-module map between direct summands is also a $B$-module map.*

*Proof.* Let $N$ be a direct summand. For $n \in N$ and $b \in B$, writing $\pi$ for the projection of $M$ onto $N$, we have $\pi bn = b\pi n = bn$, so $bn$ is in $N$. If $\varphi$ is an $A$-module map from $N$ to a direct summand $N'$, we have $\varphi b_N = \varphi b\pi = \varphi\pi b = b\varphi\pi = b_{N'}\varphi$, so $\varphi$ is a $B$-map. Setting $N' = N$, we get the second assertion. $\square$

In general the map $b \mapsto b_N$ from the bicommutant of $M$ to that of a direct summand $N$ is neither injective nor surjective.

[sss:aisb]

5.3.1. Some modules $M$ for which $A_M = B_M$. The following lemma gives a criterion under which we can deduce $A_M = B_M$ from the corresponding equality for a submodule.

[l:bfactor]

**Lemma 5.5.** *Let $M = N \oplus P$, and suppose that $P$ is a sum of submodules isomorphic to quotients of $N$. Then, the homomorphism $b \mapsto b_N$ from $B$ to the bicommutant $B_N$ of $N$ is injective. If, furthermore, $B_N$ equals the ring $A_N$ of homotheties, then $B = A_M$.*

*Proof.* By the hypothesis, $N$ generates the opposite module $M$. So any homomorphism of the opposite module is determined by its restriction to $N$. In particular, $b_N$ determines $b$. So we have on the one hand $A_M \simeq A_N \subseteq B_N$ and on the other $A_M \subseteq B_M \hookrightarrow B_N$. The second assertion should now be clear. $\square$

We conclude that the ring of homotheties equals the bicommutant for

- a free module of finite rank.
- a finitely generated module over a principal ring (because there exists a sequence of ideals $\mathfrak{a}_1 \subseteq \ldots \subseteq \mathfrak{a}_n$ such that $M \simeq A/\mathfrak{a}_1 \oplus \cdots \oplus A/\mathfrak{a}_n$.).

As special cases respectively of the above items we have:

- The centre of a matrix ring over a field consists of the scalar matrices.
- Let $u$ be an endomorphism of a finite dimensional vector space over a field $k$. An endomorophism that commutes with all endomorphisms that commute with $u$ is expressible as a polynomial in $u$ with coefficients in $k$.

[sss:bdsum]

5.3.2. Bicommutants commute with arbitrary direct sums.

[l:bdsum]

**Lemma 5.6.** *We can consider $M$ as a module for its bicommutant $B$. Let $B_{M^{\oplus I}}$ denote the ring of homotheties of a direct sum of $I$ copies of the $B$-module $M$. Then $B_{M^{\oplus I}}$ is the bicommutant of the ring of homotheties $A_{M^{\oplus I}}$ of $M^{\oplus I}$ as an $A$-module. In particular, the natural map $B_{M^{\oplus I}} \to B_{M_i}(\simeq B_M)$ for any direct summand $M_i$ $(i \in I)$ is a bijection.*

*Proof.* Think of $I \times I$ column-finite matrices with entries in $\mathfrak{L}$. The centralizer of these are scalar matrices, scalars being elements of $B$. $\square$

[ss:setupcommiso]

5.4. **Set up for considering commutation in isotypic modules.** We fix an $A$-module $F$ and let $C$ denote its commutant. Let $\mathfrak{R}$ denote the category of right $C$-modules and $\mathfrak{G}$ the category of left $A$-modules. In the sequel, we will denote by $V$ and $M$ respectively a typical object $\mathfrak{R}$ and $\mathfrak{G}$.

We define covariant functors $S$ and $T$ between these categories:

- $S : \mathfrak{R} \to \mathfrak{G}$ by $S := \cdot \otimes_C F$.
- $T : \mathfrak{G} \to \mathfrak{R}$ by $T := \mathrm{Hom}_A(F, \cdot)$.

The $A$-module structure on $V \otimes_C F$ is given by $a(v \otimes f) := v \otimes af$; the $C$-module structure on $\mathrm{Hom}_A(F, M)$ is given by $(\phi c)(f) := \phi(cf)$.

We have natural transformations:

- $\alpha$ from the identity functor on $\mathfrak{R}$ to $T \circ S$: $\alpha_V : V \to \mathrm{Hom}_A(F, V \otimes_C F)$ by $\alpha(v) = v \otimes \cdot$ , i.e., $(\alpha(v))(f) = v \otimes f$.
- $\beta$ from $S \circ T$ to the identity functor on $\mathfrak{G}$: $\beta_M : \mathrm{Hom}_A(F, M) \otimes_C F \to M$ by $\beta_M(\phi \otimes f) := \phi(f)$.

[ss:commiso]

5.5. **Commutation in isotypic modules.** Let notation be as in the previous section. In the situation when $F$ is finitely generated (as an $A$-module), in particular, when $F$ is a simple, the operations introduced in the previous section enable us to express isotypic $F$-modules and homomorphisms between them in a coordinate free way, i.e., without using specific decompositions into direct sums of copies of $F$.

[t:commiso]

**Theorem 5.7.** *Notation is fixed as in the previous section. Assume that $F$ is finitely generated as an $A$-module.*

(1) *Let $V$ be a free $C$-module with basis $(v_i)_{i \in I}$. Then $S(V) = V \otimes_C F$ is isotypic of type $F$, a direct sum of the submodules $v_i C \otimes_C F$, and the map $\alpha_V : V \to T \circ S(V)$ is a bijection; for any $V'$ in $\mathfrak{R}$, the map $S(V', V) : \mathrm{Hom}_C(V', V) \to \mathrm{Hom}_A(V' \otimes_C F, V \otimes_C F)$ is a bijection.*

(2) *Conversely, suppose $M$ is isotypic of type $F$, and let $(u_i)_{i \in I}$ be a family of injective homomorphisms of $F$ into $M$ such that $M$ is a direct sum of its submodules $u_i F$. Then $T(M) = \mathrm{Hom}_A(F, M)$ is a free $C$-module with basis $(u_i)$, and the map $\beta_M : S \circ T(M) \to M$ is a bijection; for any $M'$ in $\mathfrak{G}$, the map $T(M, M') : \mathrm{Hom}_A(M, M') \to \mathrm{Hom}_C(\mathrm{Hom}_A(F, M), \mathrm{Hom}_A(F, M'))$ is a bijection.*

PROOF: The proof (details of which we omit) is based on the following two observations:

- For $V$ in $\mathfrak{R}$ which is free, we have $\mathrm{Hom}_A(F, V \otimes_C F) \simeq V \otimes_C \mathrm{Hom}_A(F, F)$ (by the finite generation of $F$), and then $V \otimes_C \mathrm{Hom}_A(F, F) = V \otimes_C C \simeq V$.
- Let $P$ be in $\mathfrak{R}$, $Q$ and $R$ in $\mathfrak{G}$. Assume that $Q$ has the structure of a left $C$-module too and that the actions of $A$ and $C$ commute. Then we have, by 'Hom-tensor adjointness':

$$\mathrm{Hom}_A(P \otimes_C Q, R) \simeq \mathrm{Hom}_C(P, \mathrm{Hom}_A(Q, R))$$

**Remark 5.8.** *In the above theorem, we could omit the hypothesis that $F$ is finitely generated, but in its place restrict ourselves to free finitely generated modules over $C$ on the one hand and finite direct sums of copies of $F$ on the other.*

[c:1:t:commiso]

**Corollary 5.9.** *Let $F$ be finitely generated and $V$ be a free $C$-module. Then the rings* $\mathrm{End}_C(V)$ *and* $\mathrm{End}_A(V \otimes_c F)$ *are isomorphic.*

[c:2:t:commiso]

**Corollary 5.10.** *Let $F$ be finitely generated and $M$ be isotypic of type $F$. Then the rings* $\mathrm{End}_A(M)$ *and* $\mathrm{End}_C(\mathrm{Hom}_A(F, M))$ *are isomorphic.*

If $M$ is a finite direct sum of $n$ copies of $F$, then $\mathrm{End}_A(F)$ is isomorphic to the ring $M_n(C)$ of $n \times n$ matrices with entries in the ring $C$.

[c:3:t:commiso]

**Corollary 5.11.** *Let $M$ be isotpyic of type $F$, and let $V = \mathrm{Hom}_A(F, M)$. Suppose that $F$ is finitely generated and identify $M$ with $V \otimes_C F$ via $\beta_M$. Then the direct factors $V'$ of $V$ that are free $C$-modules, and the direct factors $M'$ of $M$ that are $F$-isotypic are in bijective correspondence by the formulas $M' = V' \otimes_C F$ and $V' = \mathrm{Hom}_A(F, M')$.*

[x:comm]

5.6. **Exercises.**

5.6.1. L. et $M$ be a vector space over a division ring $D$ of finite dimension $d$. Then the commutant $C := \mathrm{End}_D M$ is isomorphic to the ring $M_d(D^{\mathrm{opp}})$ of $d \times d$ matrices with entries in $D^{\mathrm{opp}}$.