[s:anmod]

6. Artinian and Noetherian modules

[ss:anmoddef]

6.1. **Definitions and elementary properties.** A module is *Artinian* (respectively *Noetherian*) if it satisfies either of the following equivalent conditions:

- every non-empty collection of submodules contains a minimal (repsectively maximal) element with respect to inclusion.
- any descending (respectively ascending) chain of submodules stabilises.

A module is Artinian (respectively Noetherian) if and only if it is so over its ring of homotheties.

An infinite direct sum of non-zero modules is neither Artinian nor Noetherian. A vector space is Artinian (respectively Noetherian) if and only if its dimension is finite.

We now list some elementary facts about Artinian and Noetherian modules. The following would continue to be true if we replaced 'Artinian' by 'Noetherian':

- Submodules and quotient modules of Artinian modules are Artinian.
- If a submodule N of a module M and the quotient M/N by it are Artinian, then so is M.
- A finite direct sum is Artinian if and only if each of the summands is so.

A comment about the simultaneous presence of the both conditions:

- A module is both Artinian and Noetherian if and only if it has finite length.

Pertaining to the Noetherian condition alone, we make two more observations:

- Any subset S of a Noetherian module contains a finite subset that generates the same submodule as S.
- A module is Noetherian if and only if every submodule of it is finitely generated.

[s:fittingkrs]

6.2. **Decomposition into indecomposables of a finite length module.** Let u be an endomorphism of a module M. We have

$$0 \subseteq \operatorname{Ker} u \subseteq \operatorname{Ker} u^2 \subseteq \operatorname{Ker} u^3 \subseteq \dots$$
$$M \supseteq \operatorname{Im} u \supseteq \operatorname{Im} u^2 \supseteq \operatorname{Im} u^3 \supseteq \dots$$

Suppose that the ascending chain above stabilises (e.g., when M is Noetherian), say $\operatorname{Ker} u^n = \operatorname{Ker} u^{n+1}$. Then $\operatorname{Ker} u^n \cap \operatorname{Im} u^n = 0$. Indeed if and $u^n x = 0$ and $x = u^n y$, then $u^{2n} y = 0$, so $y \in \operatorname{Ker} u^{2n} = \operatorname{Ker} u^n$ and $x = u^n y = 0$. If u were also surjective, then so would be u^n, which means $\operatorname{Im} u^n = M$, and so $\operatorname{Ker} u^n = 0$, which means u^n (and so also u) is injective. Thus:

(6.1) A surjective endomorphism of a Noetherian module is bijective.

Suppose that the descending chain above stablises (e.g., when M is Artinian), say $\operatorname{Im} u^n = \operatorname{Im} u^{n+1}$. Then $M = \operatorname{Ker} u^n + \operatorname{Im} u^n$. Indeed, for $x \in M$, choosing y such that $u^n x = u^{2n} y$, we have $x = (x - u^n y) + u^n y$. If u were also injective, then so would be u^n, which means $\operatorname{Ker} u^n = 0$, so $M = \operatorname{Im} u^n$, which means u^n (and so also u) in surjective. Thus:

(6.2) An injective endomorphism of an Artinian module is bijective.

Suppose now that M is of finite length (equivalently, both Noetherian and Artinian). Then the above considerations show that for sufficiently large n we have a direct sum decomposition

$$M = \operatorname{Ker} u^n \oplus \operatorname{Im} u^n \tag{6.3}$$

If M were also indecomposable, then either $\operatorname{Ker} u^n = M$, in which case u is nilpotent, or $\operatorname{Ker} u^n = 0$ and $\operatorname{Im} u^n = M$, in which case u^n (and so also u) is invertible, which proves the first half of the following

[p:local]

Proposition 6.1. *The non-invertible endomorphisms of an indecomposable module M of finite length are nilpotent and form a two sided ideal.*

Proof. The first half having already been proved, we need only prove the second half. For a nilpotent endomorphism u, and φ any endomorphism, φu and $u\varphi$ are non-invertible, and so nilpotent. Now suppose u and v are nilpotent endomorphisms. Suppose $u + v$ is not nilpotent. Then it is invertible. Let φ be such that $\varphi(u+v) = 1$. Writing $\varphi u = 1-\varphi v$, we observe that φu is on the one hand nilpotent and on the other invertible. □

[t:krs]

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Theorem 6.2. *A module of finite length is a finite direct sum of indecomposable submodules. Further, any two such decompositions with no trivial factors are the same, i.e., the components are respectively isomorphic after a permutation.*

Proof. The decomposition into a finite direct sum of indecomposable submodules follows easily by an induction on the length. We will now prove the uniqueness. Suppose $\oplus_{i=1}^{m} M_i$ and $\oplus_{i'=1}^{m'} M'_{i'}$ are two such decompositions of a module M. We prove the following claim by induction and that will suffice:

> for $0 \leq j \leq m$ there exists an automorphism α_j of M such that, after a possible rearrangement of the M_i, we have $\alpha_j M'_k = M_k$ for $1 \leq k \leq j$.

The base case of the induction ($j = 0$) is vacuous: we can take α_0 to be the identity. Now, assuming the statement for some $j-1 < k$, we will prove it for j. Writing $\alpha_{j-1} M'_{i'} =: M''_{i'}$, consider the decomposition $\oplus_{i'=1}^{m'} M''_{i'}$. We have $M''_k = M_k$ for $1 \leq k < j$.

Let p_k, p'_k, and p''_k denote respectively the projection onto M_k, M'_k, and M''_k with respect to the respective decompositions. The restriction to M''_j of the projection p''_j is of course the identity but it also equals $\sum_k p''_j p_k$. By the previous proposition, there exists a $1 \leq k \leq n$ such that $p''_j p_k$ is an automorphism of M''_j. We claim that $j \leq k$. Indeed, if $k < j$, then, since $p_k M''_j \subseteq M_k = M''_k$, we have $p''_j p_k M''_j = 0$, a contradiction, and the claim is proved.

After a rearrangement of the M_k if necessary, we can take $k = j$. The automorphism α_j is now defined as $\varphi\alpha_{j-1}$ where φ is the endomorphism of M that is identity on all M''_l except $l = j$ and is p_j on M''_j: $\varphi := 1 - p''_j + p_k p''_j$. We claim that φ is injective. It follows from the claim and what has been said earlier in this subsection that φ is bijective and that α_j is an automorphism. To prove the claim, suppose that $\varphi x = 0$. Write $x - p''_j x = -p_k p''_j x \in M_k$; we see that $p''_j p_k p''_j x = 0$ since p''_j clearly kills the left side. But $p''_j p_k$ being an automorphism of M''_j, we conclude that $p''_j x = 0$, so $0 = \varphi x = x$, and the claim is proved.

It remains only to show that $\alpha_j M'_k = M_k$ for $1 \leq k \leq j$. This is evident for $k < j$: indeed, $\varphi\alpha_{j-1} M'_k = \varphi M''_k = M''_k = M_k$. We now prove $\alpha_j M'_j = M_j$. Since $\alpha_j M'_j = \varphi M''_j \subseteq M_j$, it follows that $M_j = \varphi M''_j \oplus (M_j \cap \sum_{k\neq j} \varphi M''_k)$, so $M_j = \varphi M''_j = \alpha_j M'_j$ by the indecomposability of M_j. □

[ss:anrings]

6.3. **Artinian and Noetherian rings.** A ring is *left Artinian* (respectively *left Noetherian*) if it is so as a left module over itself. *Right Artinian* and *right Noetherian* rings are similarly defined. We will omit the adjective 'left' and just say Artinian (respectively Noetherian) to mean left Artinian (respectively left Noetherian).[7]

- If A contains a subring (with unity) which is a division ring, and if A is finite dimensional as a left module over the division ring, then A is Artinian and Noetherian.
- A principal ideal ring is Noetherian.
- We will see later than Artinian rings are Noetherian. So, if a ring is Artinian, then ${}_AA$ has finite length, called the *left length* (just *length* if A is commutative).

Artinian rings are Noetherian.

[p:anring]

Proposition 6.3. *Let M be a faithful A-module whose opposite is finitely generated, say by $\{m_1, \ldots, m_n\}$. Then the map $a \mapsto (am_1, \ldots, am_n)$ defines an injection of A-modules from ${}_AA$ into $M^{\oplus n}$.*

Proof. The map is clearly A-linear. If $am_1 = \ldots = am_n = 0$, then since the m_i generate the opposite M, it follows that $aM = 0$, so $a = 0$ by faithfulness. $\square$

The following properties are elementary to prove. They are stated for Artinian rings but the corresponding statements hold also for Noetherian rings.

- A finite direct product of Artinian rings is Artinian.
- A quotient of an Artinian ring (by a two sided ideal) is Artinian.
- A finitely generated module over an Artinian ring is Artinian.
- A ring is Artinian if it admits a faithful Artinian module whose opposite is fintiely generated. (This follows from the proposition above.)

The following statement is made only for Noetherian rings:

- A commutative ring that admits a faithful Noetherian module is Noetherian.

[7] There are rings that are left Artinian but not right Artinian. Similarly for Noetherian.