Annals of Mathematics, **106** (1977), 295-308

Tensor products of finite and infinite dimensional representations of semisimple Lie groups

By GREGG ZUCKERMAN*

I.M. Gelfand and V.A. Ponomarev initiated in [4] (see also [5], [6]) the study of the category of Harish-Chandra modules over a real semisimple Lie algebra equipped with a Cartan involution. In this paper we prove some very general theorems about tensor products of finite dimensional modules with Harish-Chandra modules. These theorems yield new structural information about the category of Harish-Chandra modules, as well as new information on the classification, construction, and properties of the infinite dimensional irreducible Harish-Chandra modules.

We study the Harish-Chandra modules associated to a connected semisimple Lie group G having finite center. We let $\mathfrak{g}_0$ be the Lie algebra of G, and $\mathfrak{k}_0$ be the Lie algebra of a maximal compact subgroup K in G. Let $\mathcal{G}$ be the (complex) universal enveloping algebra of $\mathfrak{g}_0$. Lepowsky has defined in [21] the notion of a compatible $(\mathcal{G}, K)$-module (see Section 2 for the exact definition). By definition, a Harish-Chandra $(\mathcal{G}, K)$-module is finitely generated over $\mathcal{G}$ and has each K-isotypic subspace finite dimensional. Let $\mathcal{A} = \mathcal{A}(\mathcal{G}, K)$ be the category of Harish-Chandra modules.

Irreducible modules in $\mathcal{A}$ arise from irreducible quasisimple continuous representations of G on Banach spaces (see [7], [8]). More general modules in $\mathcal{A}$ arise from inducing irreducible finite dimensional representations of a parabolic subgroup of G. Gelfand and Ponomarev [5] classify all Harish-Chandra modules over the Lie algebra of the Lorentz group. An analogous classification for a general G appears to be impossible [6]. Nevertheless, our results demonstrate a kind of periodicity (clearly visible in the results of [5]) in the category $\mathcal{A}$ (see Theorem 1.2, part ②).

This periodicity has many applications. Lemma 5.4 below has already proven to be of crucial importance in various investigations of the discrete series of square-integrable representations of G (see [18], [26], [28]). This lemma, in combination with Theorem 1.3 below, now leads to the simplest and most efficient construction and characterization of the so-called "limits" of discrete series (see Theorem 5.7), which have turned out to be essential

* Work partially supported by NSF Grant # MCS 76-05921.

ingredients in the recently announced [19] classification of the irreducible tempered representations of G.

For other work on tensor products see [17], [20], and [22].

1. Periodicity

We describe in this section our basic constructions and results in functorial terms. Let F be a finite dimensional irreducible $\mathfrak{g}_0$ module. We will assume (by passing to a finite covering group of G) that any such F lifts to a representation of G. If $A \in \mathfrak{A}$, so is $A \otimes F$ (i.e. $A \otimes_{\mathbf{C}} F \in \mathfrak{A}$) (see Lemma 2.1). $(\ ) \otimes F$ is an exact functor on the abelian category $\mathfrak{A}$. Let $\mathfrak{Z}$ be the center of $\mathfrak{G}$ and let λ be a character of $\mathfrak{Z}$, i.e. $\lambda \in \mathrm{Hom}(\mathfrak{Z}, \mathbf{C})$. We can define a projection functor p_λ from $\mathfrak{A}$ to $\mathfrak{A}$: for $A \in \mathfrak{A}$, $p_\lambda A$ is the maximal submodule of A on which $z - \lambda(z) \cdot 1$ is nilpotent for every $z \in \mathfrak{Z}$. By Lemma 2.2, p_λ is an exact functor; moreover, for $A \in \mathfrak{A}$ we have $A = \bigoplus_\lambda p_\lambda A$, where the sum is actually finite. We can decompose $\mathfrak{A}$ into subcategories $\mathfrak{A}_\lambda = p_\lambda \mathfrak{A}$ (see [4]).

It makes sense now to decompose $(\ ) \otimes F$ via the projections p_λ: i.e. we can form the composite functors $p_{\lambda_1} \circ [(\ ) \otimes F] \circ p_{\lambda_2}$, with λ_1 and λ_2 characters of $\mathfrak{Z}$. Our main results concern certain of these composites.

We need an explicit parameterization of $\mathrm{Hom}(\mathfrak{Z}, \mathbf{C})$. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$ and $\mathfrak{h}$ any Cartan subalgebra in $\mathfrak{g}$. Let $\mathfrak{h}^*$ be the dual of $\mathfrak{h}$, and W be the Weyl group of $(\mathfrak{h}, \mathfrak{g})$. Let $\mathfrak{C}_0$ be a closed Weyl chamber in the real span of the roots of $(\mathfrak{h}, \mathfrak{g})$. By the Harish-Chandra homomorphism [10], $\mathfrak{Z}$ can be identified with a particular fundamental domain $\mathfrak{C}$ for the action of W on $\mathfrak{h}^*$ (see Def. 2.3). The closure of $\mathfrak{C}$ is a tube over $\mathfrak{C}_0$.

We let $W(\lambda)$ be the stabilizer in W of $\lambda \in \mathfrak{C}$; λ is regular if $W(\lambda)$ is trivial; interior points of $\mathfrak{C}$ are regular. Boundary points of $\mathfrak{C}_0$ are singular.

We denote by μ the highest weight in $\mathfrak{C}_0$ of the finite dimensional module F, and we write $F = F^\mu$. The module contragredient to F^μ has lowest weight $-\mu$ and will be denoted $F_{-\mu}$. A form $\mu \in \mathfrak{C}_0$ arises from some F if and only if μ is integral with respect to the simple coroots that define the walls of $\mathfrak{C}_0$.

Definition 1.1. For $\lambda \in \mathfrak{C}$ and μ integral in $\mathfrak{C}_0$, let

a) $\varphi^\lambda_{\lambda+\mu} = p_{\lambda+\mu} \circ [(\ ) \otimes F^\mu] \circ p_\lambda$.

b) $\psi^{\lambda+\mu}_\lambda = p_\lambda \circ [(\ ) \otimes F_{-\mu}] \circ p_{\lambda+\mu}$.

THEOREM 1.2. *Given λ and μ as in the definition:*

① *If $A \in \mathfrak{A}_\lambda$ and $\varphi^\lambda_{\lambda+\mu} A = 0$, then $A = 0$.*

② *If $W(\lambda + \mu) = W(\lambda)$, $\varphi^\lambda_{\lambda+\mu}$ restricts to an isomorphism of $\mathfrak{A}_\lambda$ with $\mathfrak{A}_{\lambda+\mu}$; $\psi^{\lambda+\mu}$ is a natural inverse, in the sense that $\psi\varphi$ is naturally isomorphic* (*see* [23]) *to the identity functor on* $\mathcal{A}_\lambda$, *and* $\varphi\psi$ *is naturally isomorphic to the identity functor on* $\mathcal{A}_{\lambda+\mu}$. *If* $A \in \mathcal{A}_\lambda$ *is irreducible,* $\varphi^\lambda_{\lambda+\mu}A$ *is irreducible. If* $B \in \mathcal{A}_{\lambda+\mu}$ *is irreducible,* $\psi^{\lambda+\mu}_\lambda B$ *is irreducible.*

It can happen that for nonzero $B \in \mathcal{A}_{\lambda+\mu}$, $\psi^{\lambda+\mu}_\lambda B = 0$ (in this case, λ is necessarily singular). Nevertheless, we have some control over ψ. Any Harish-Chandra module A has a composition series of finite length $l(A)$ depending only on A (see [21]). We say that $A \in \mathcal{A}$ is primary, or that A is a multiple of an irreducible module, if all the composition factors of A are mutually equivalent (A need not be completely reducible). The module "0" is certainly primary.

THEOREM 1.3. ① ψ *maps primary modules to primary modules.*

② *Given an irreducible module* $A \in \mathcal{A}_\lambda$: *every composition factor* B *of* $\varphi^\lambda_{\lambda+\mu}A$ *must map under* $\psi^{\lambda+\mu}_\lambda$ *to a multiple of* A; *in particular, at least one composition factor* B_1 *of* φA *maps to a nonzero multiple of* A.

③ *Conversely, if an irreducible module* $B \in \mathcal{A}_{\lambda+\mu}$ *maps under* ψ *to a nonzero multiple of* $A \in \mathcal{A}_\lambda$, *then* B *is a composition factor of* φA.

To conclude this section we mention (see Lemma 5.4) that if G has a discrete series, the φ functors map the discrete series into itself. The ψ functors enlarge the discrete series to include the limits of discrete series (see Theorem 5.7).

2. Preliminaries on φ and ψ

For completeness we state here the definition of a compatible $(\mathcal{G}, K)$-module V (see [21]).

(i) V is a $\mathcal{G}$-module.

(ii) V is a K-module such that every vector $v \in V$ is K-smooth and generates a finite dimensional K-stable subspace.

(iii) If $\mathcal{K}$ is the enveloping algebra of $\mathfrak{k}$, the $\mathcal{K}$-module derived from the smooth representation of K coincides with the restriction of the $\mathcal{G}$-module structure to $\mathcal{K}$.

(iv) If $k \in K$, $x \in \mathcal{G}$, and $v \in V$, then

$$k \cdot (x \cdot v) = [\mathrm{Ad}(k)x] \cdot (k \cdot v) .$$

If V is a Harish-Chandra $(\mathcal{G}, K)$-module and W is an irreducible (finite dimensional) K-module, then we have $\dim \mathrm{Hom}_K(W, V) < \infty$. For any compatible $(\mathcal{G}, K)$-module V, let V^W be the K-submodule of vectors each generating a K-module isomorphic to W. Then, as a K-module, $V = \bigoplus_{\{W\}} V^W$ where

$\{W\}$ denotes the equivalence class of W.

Now let F be the finite dimensional $\mathcal{G}$-module of Section 1.

LEMMA 2.1. *If* $A \in \mathfrak{A}$, *then* $A \otimes F \in \mathfrak{A}$.

Proof (see [3]). Let W be an irreducible K-module. Then

$$\mathrm{Hom}_K(W, A \otimes F) \cong \mathrm{Hom}_K(W \otimes F^{\vee}, A) ,$$

where $F^{\vee} = \mathrm{Hom}_{\mathrm{C}}(F, \mathfrak{C})$, the module contragredient to F. $W \otimes F^{\vee}$ decomposes into a finite sum of irreducible K-modules. Hence, $\dim \mathrm{Hom}_K(W, A \otimes F) < \infty$.

On the other hand, since A is finitely generated over $\mathcal{G}$ and F is finite dimensional, $A \otimes F$ is also finitely generated over $\mathcal{G}$.

We next prove the main properties of the projection functors p_λ.

LEMMA 2.2. *If* $A \in \mathfrak{A}$, $p_\lambda A$ *is a direct summand of* A, *and* p_λ *is an exact functor from* $\mathfrak{A}$ *to* $\mathfrak{A}$.

Proof. Let I_λ be the kernel of $\lambda: \mathcal{Z} \to C$. Let $\mathcal{Z}_\lambda$ be the localization (see [2]) of $\mathcal{Z}$ at I_λ. Map $p_\lambda A$ to $A \otimes_{\mathcal{Z}} \mathcal{Z}_\lambda$ by sending a to $a \otimes 1$. This map—call it β—is a $(\mathcal{G}, K)$-module morphism. By the finite dimensionality of the K-isotypic space A^W, the restriction of β to $(p_\lambda A)^W$ induces an isomorphism with $A^W \otimes_{\mathcal{Z}} \mathcal{Z}_\lambda$. Hence, β is an isomorphism.

Thus, $p_\lambda A$ is a direct summand of A. Moreover, by [2, Chapter II], p_λ is exact.

We conclude this section by describing the subset $\mathfrak{C}$ of $\mathfrak{h}^*$ that parameterizes $\mathrm{Hom}(\mathcal{Z}, \mathbf{C})$. We have chosen a closed Weyl chamber $\mathfrak{C}_0$ in the real span $h^*(\mathbf{R})$ of the roots of $\mathfrak{h}$ in $\mathfrak{g}$. We write $\mathfrak{h}^* = \mathfrak{h}^*(\mathbf{R}) + \sqrt{-1}\,\mathfrak{h}^*(\mathbf{R})$, and for $\lambda \in \mathfrak{h}^*$, we write $\lambda = \mathrm{Re}\,\lambda + \sqrt{-1}\,\mathrm{Im}\,\lambda$, with $\mathrm{Re}\,\lambda$ and $\mathrm{Im}\,\lambda \in \mathfrak{h}^*(\mathbf{R})$.

First, construct the tube $\tilde{\mathfrak{C}} = \{\lambda \in \mathfrak{h}^* \mid \mathrm{Re}\,\lambda \in \mathfrak{C}_0\}$. Then, for each $\lambda \in \tilde{\mathfrak{C}}$, let Δ_λ be the set of $\mathfrak{C}_0$-simple roots α such that $(\mathrm{Re}\,\lambda \mid \alpha) = 0$.

Definition 2.3. $\mathfrak{C} = \{\lambda \in \tilde{\mathfrak{C}} \mid \alpha \in \Delta_\lambda \text{ implies } (\mathrm{Im}\,\lambda \mid \alpha) \geqq 0\}$.

LEMMA 2.4. $\mathfrak{C}$ *is a fundamental domain for the action of* W *on* $\mathfrak{h}^*$.

Remarks. ①: $\mathfrak{C}$ is a semigroup. In particular, if λ is an element of $\mathfrak{C}$ and $\mu \in \mathfrak{C}$ is integral, then $\lambda + \mu \in \mathfrak{C}$.

②: Given $\mathfrak{C}$, we can now define φ and ψ as in Definition 1.1. This definition will not depend on the choice of $\mathfrak{h}$ or the order on the roots of $\mathfrak{h}$.

Proof of Lemma 2.4. We have $\mathfrak{h}^* = W\tilde{\mathfrak{C}}$. For any $\lambda \in \tilde{\mathfrak{C}}$, let $\tilde{\mathfrak{C}}(\lambda) = \{\lambda' \mid \mathrm{Re}\,\lambda' = \mathrm{Re}\,\lambda\}$. The reflections s_α with $\alpha \in \Delta_\lambda$ generate the Weyl group $W(\mathrm{Re}\,\lambda)$, which operates on $\tilde{\mathfrak{C}}(\lambda)$ with fundamental domain

$$\{\lambda' \in \tilde{\mathfrak{C}}(\lambda) \mid \alpha \in \Delta_\lambda \text{ implies } (\operatorname{Im} \lambda' | \alpha) \geqq 0\}\,.$$

So $\mathfrak{h}^* = W\mathfrak{C}$. Arguing first with real parts, then with imaginary parts, we see that no two elements of $\mathfrak{C}$ lie in the same W-orbit.

3. The composite functors, $\psi\varphi$ and $\varphi\psi$

We prove the following key lemma:

LEMMA 3.1. *If A in $\mathfrak{A}_\lambda$ is irreducible, $\psi\varphi A$ is a multiple of A, and* $l(\psi\varphi A) = m \overset{\text{def.}}{=} [W(\lambda)\colon W(\lambda + \mu)]$.

Reminder. A multiple B of A need not be completely reducible; we only require all composition factors of B to be isomorphic to A.

From the proof of Lemma 3.1 we will easily see

LEMMA 3.2. *If $W(\lambda) = W(\lambda + \mu)$ and $B \in \mathfrak{A}_{\lambda+\mu}$ is irreducible, then* $\varphi\psi B \cong B$.

Our method of proof will be global character theory (see [9], [10], [16]). By the subquotient theorem ([8], [21]), every irreducible Harish-Chandra module A is isomorphic to the derived module of some "global" representation π of G. The global character θ_π of π will depend only on the isomorphism class of A, so that we write $\theta(A) \overset{\text{def.}}{=} \theta_\pi$ (see [9]). Moreover, the isomorphism class of A is completely determined by $\theta(A)$. If $A_1, \cdots, A_r$ is a set of mutually inequivalent irreducible Harish-Chandra modules, the characters $\theta(A_1), \cdots, \theta(A_r)$ will be linearly independent (see [9]).

If C is a reducible Harish-Chandra module, we define $\theta(C)$ to be the sum of the characters of the composition factors for a composition series of C; this sum will be independent of the choice of composition series. If we have an exact sequence $0 \to C' \to C \to C'' \to 0$, we will have $\theta(C) = \theta(C') + \theta(C'')$. Finally, if $\theta(C_1) = \theta(C_2)$, then in some order the composition factors of C_1 are isomorphic to the composition factors of C_2.

The exactness of φ, in combination with the above discussion on characters, implies that for $A \in \mathfrak{A}_\lambda$, $\theta(\varphi A)$ depends only on $\theta(A)$. In addition we conclude that Lemma 3.1 is equivalent to

LEMMA 3.3. *If $A \in \mathfrak{A}_\lambda$, $\theta(\psi\varphi A) = m\theta(A)$.*

Before proving Lemma 3.3 we need to discuss a simple formula for $\theta(F \otimes A)$, F a finite dimensional G-module. $\theta(F)$ is a C^∞ function on G. Hence, it makes sense to write the product $\theta(F)\theta(A)$.

LEMMA 3.4. $\theta(F \otimes A) = \theta(F)\theta(A)$.

Lemma 3.4 follows easily from the distributional definition of $\theta(A)$ (see

[9]). We now review the crucial results from character theory that make possible the application of Lemma 3.4:

① $\theta(A)$ is representable by a function—which we also call $\theta(A)$—which is locally summable with respect to Haar measure [11]. Thus, if we know $\theta(A)$ on the set G' of regular semisimple elements, we know $\theta(A)$ completely. Lemma 3.4 can now be reinterpreted in terms of multiplying functions pointwise.

② $\theta(A)$ is real analytic on G' and has there a rather explicit form generalizing the Weyl character formula. Every element in G' lies in a unique Cartan subgroup, so that we need to describe the restriction of $\theta(A)$ to $H' = H \cap G'$, H a Cartan subgroup of G. We fix an order on the roots of $\mathfrak{h}$ in $\mathfrak{g}$, $\mathfrak{h}$ the complexification of the Lie algebra $\mathfrak{h}_0$ of H. Let $\mathfrak{C}_0$ be the corresponding Weyl chamber, and $\mathfrak{C}$ be as in Definition 2.3. By assumption on G we can define a Weyl denominator ∇ on H (see [10]). We may also suppose that $\mathfrak{h}_0 = \mathfrak{h}_- + \mathfrak{h}_+$, where $\mathfrak{h}_- = \mathfrak{h}_0 \cap \mathfrak{k}_0$ and $\mathfrak{h}_+ = \mathfrak{h}_0 \cap \mathfrak{p}_0$ ($\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ is the Cartan decomposition of $\mathfrak{g}_0$). Then, if we set $H_- = H \cap K$, we have $H = H_- \exp \mathfrak{h}_+$.

Let κ be $\nabla \cdot \theta(A)|H'$. If α is a root of $\mathfrak{h}$ in $\mathfrak{g}$, let ξ_α be the character of H defined by $\mathrm{ad}(h)X_\alpha = \xi_\alpha(h)X_\alpha$, where X_α is a root vector for α. Finally, let $H'(\mathbf{R}) = \{h \in H | \xi_\alpha(h) \neq 1$ if α is a real root of $\mathfrak{h}\}$. Then κ extends to an analytic function on $H'(\mathbf{R})$.

Let $\mathfrak{F}$ be a connected component of $H'(\mathbf{R})$, and $h_0 \in H_-$ be an element on the boundary of $\mathfrak{F}$. Then if $A \in \mathfrak{A}_\lambda$,

$$\kappa(h_0 \exp X) = \sum_{s \in W} p_s(X) \exp\{s\lambda(X)\}\ , \tag{3.5}$$

if $X \in \mathfrak{h}_0$ is sufficiently small and $h_0 \exp X \in \mathfrak{F}$, where the functions p_s are polynomials on $\mathfrak{h}$. (For background see [10], [16].)

③ If F is a finite dimensional module and $P(F)$ is the set of weights, counted with their multiplicities, of F with respect to $\mathfrak{h}$, then

$$\theta(F)(h_0 \exp X) = \sum_{\nu \in P(F)} \xi_\nu(h_0) \exp\{\nu(X)\}\ . \tag{3.6}$$

Here, ξ_ν is the character of H corresponding to a weight vector f_ν in F: $\xi_\nu(h)f_\nu = h \circ f_\nu$.

④ If $E \in \mathfrak{A}$, we can consider the functions $\kappa = \nabla \cdot \theta(E)$ defined for each Cartan subgroup; let Λ be the set of $\lambda \in \mathfrak{C}$ such that $p_\lambda E \neq 0$. Then we will have

$$\kappa(h_0 \exp X) = \sum_{\lambda \in \Lambda} \sum_{s \in W} p_{s,\lambda}(X) \exp\{s^\lambda(X)\}\ .$$

Moreover, $\nabla \cdot \theta(p_\lambda E)(h_0 \exp X)$ will be given by

$$\sum_{s \in W} p_{s,\lambda}(X) \exp\{s\lambda(X)\}\ .$$

Proof of Lemma 3.3. We take $A \in \mathfrak{A}_\lambda$ and write $\varphi = \varphi^\lambda_{\lambda+\mu}$. We have

$$\nabla \cdot \theta(F^\mu)\theta(A)(h_0 \exp X) = \sum_{\nu \in P(F^\mu)} \sum_{s \in W} \xi_\nu(h_0) p_s(X) \exp\{(s\lambda + \nu)(X)\} .$$

Hence, to compute $\theta(\varphi A)$, we must determine for which $\nu \in P(F^\mu)$ there exists $t \in W$ such that $t(s\lambda + \nu) = \lambda + \mu$. Suppose we can find such a t. We have

$$ts(\operatorname{Re}\lambda) + t\nu = (\operatorname{Re}\lambda) + \mu ,$$

and

$$ts(\operatorname{Im}\lambda) = \operatorname{Im}\lambda .$$

Since $\operatorname{Re}\lambda$ is dominant, $ts(\operatorname{Re}\lambda)$ is lower than or equal to $\operatorname{Re}\lambda$. Since μ is the highest weight of F^μ and since $P(F^\mu)$ is W-stable, $t\nu$ is lower than or equal to μ. Hence, $ts(\operatorname{Re}\lambda) = (\operatorname{Re}\lambda)$, $ts\lambda = \lambda$, and $t\nu = \mu$, or $\nu = t^{-1}\mu$. Multiplying t on the left by an element of $W(\lambda + \mu)$ will not change ν. On the other hand, ts must be in $W(\lambda)$. We conclude that

(3.7)
$$\nabla \cdot \theta(\varphi A)(h_0 \exp X) = \sum_{t \in \frac{W(\lambda)}{W(\lambda+\mu)}} \sum_{s \in W} \xi_{st^{-1}\mu}(h_0) p_s(X) \exp\{st^{-1}(\lambda + \mu)(X)\} .$$

Now suppose $B \in \mathfrak{A}_{\lambda+\mu}$. We write $\psi = \psi^{\lambda+\mu}_\lambda$. We have

$$\nabla \cdot \theta(F_{-\mu})\theta(B)(h_0 \exp X) = \sum_{\nu \in P(F_{-\mu})} \sum_{s \in W} \xi_\nu(h_0) q_s(X) \exp\{[s(\lambda + \mu) + \nu](X)\} ,$$

where the q_s are polynomials in the expression for $\nabla \cdot \theta(B)$. To compute $\theta(\psi B)$, we must determine for which $\nu \in P(F_{-\mu})$ there exists $t \in W$ such that $t[s(\lambda + \mu) + \nu] = \lambda$. If t exists, then

$$(\operatorname{Re}\lambda) + \mu = s^{-1}t^{-1}(\operatorname{Re}\lambda) - s^{-1}\nu ,$$

and

$$(\operatorname{Im}\lambda) = s^{-1}t^{-1}(\operatorname{Im}\lambda) .$$

Again, $s^{-1}t^{-1}(\operatorname{Re}\lambda)$ is lower than or equal to $(\operatorname{Re}\lambda)$, and $s^{-1}\nu$ is higher than or equal to $-\mu$. Hence $ts\,\lambda = \lambda$ and $\nu = -s\mu$. We can therefore take $t = s^{-1}$. We conclude that

(3.8) $$\nabla \cdot \theta(\psi B)(h_0 \exp X) = \sum_{s \in W} \xi_{-s\mu}(h_0) q_s(X) \exp s\lambda(X) .$$

We apply the above formula to the case $B = \varphi A$. By substituting st for s in formula (3.7) and interchanging sums we find

$$\nabla \cdot \theta(\varphi A)(h_0 \exp X) = \sum_{s \in W} q_s(X) \exp s(\lambda + \mu)(X) ,$$

where

$$q_s(X) = \xi_{s\mu}(h_0) \sum_{t \in \frac{W(\lambda)}{W(\lambda+\mu)}} p_{st}(X) .$$

Hence,

$$\begin{aligned}\nabla \cdot \theta(\psi\varphi A) &= \textstyle\sum_{s \in W} \left(\sum_{t \in \frac{W(\lambda)}{W(\lambda+\mu)}} p_{st}(X)\right) \exp s\lambda(X) \\ &= [W(\lambda) : W(\lambda + \mu)] \textstyle\sum_{s \in W} p_s(X) \exp s\lambda(X) \\ &= m \cdot \nabla \cdot \theta(A) .\end{aligned}$$

Lemma 3.2 also follows from formulas (3.7) and (3.8) in the case $m = 1$. We mention that Theorem 1 in [3] asserts that for any λ and for $A \in \mathfrak{A}_\lambda$, the polynomials p_s are constants. The proof given in [3] of this theorem can in fact be restated in our framework:

A theorem of Harish-Chandra [10] says that if λ_1 is regular and $A_1 \in \mathfrak{A}_{\lambda_1}$, then the polynomials in $\nabla \cdot \theta(A_1)$ are constants. Given a singular λ, find an integral weight μ such that $\lambda + \mu = \lambda_1$ is regular. Then for $A \in \mathfrak{A}_\lambda$, the polynomials in $\nabla \cdot \theta(\varphi_\lambda^{\lambda_1}, A)$ are constants. By formula (3.7), the polynomials in $\nabla \cdot \theta(A)$ must be constants.

4. Proofs of Theorems 1.2 and 1.3

Part ① of Theorem 1.2 is an immediate corollary of formula (3.7). For part ② we need

LEMMA 4.1. *ψ is a left adjoint to φ;* i.e. *if $A \in \mathfrak{A}_\lambda$ and $B \in \mathfrak{A}_{\lambda+\mu}$,* Hom$(B, \varphi A)$ *is naturally isomorphic to* Hom$(\psi B, A)$.

Proof. Hom$(B, F^\mu \otimes A)$ is naturally isomorphic to Hom$(F_{-\mu} \otimes B, A)$, since $F_{-\mu} \otimes B$ is naturally isomorphic to Hom(F^μ, B). Only the direct summand $p_{\lambda+\mu}(F^\mu \otimes A)$ contributes to Hom$(B, F^\mu \otimes A)$; likewise only the direct summand $p_\lambda(F_{-\mu} \otimes B)$ contributes to Hom$(F_{-\mu} \otimes B, A)$. The lemma follows.

We can now define a natural transformation j of functors (see [23]) from $\psi\varphi$ to the identity functor. We take, for $A \in \mathfrak{A}_\lambda$, $j(A)$ to be the map adjoint to the identity map of φA under the isomorphism Hom$(\psi\varphi A, A) \cong$ Hom$(\varphi A, \varphi A)$. The crucial property of $j(A)$ is that it is nonzero if A is nonzero.

Proof of Theorem 1.2, *Part* ②. By hypothesis, $W(\lambda + \mu) = W(\lambda)$, so for A irreducible in $\mathfrak{A}_\lambda$, $j(A)$ is an isomorphism, by Lemma 3.1. For arbitrary C in $\mathfrak{A}_\lambda$ we prove by induction on $l(C)$ that $j(C)$ is an isomorphism: let C' be a maximal submodule of C, and $C'' = C/C'$. We have, by the naturality of j, a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \psi\varphi C' & \longrightarrow & \psi\varphi C & \longrightarrow & \psi\varphi C'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j(C')} & & \downarrow{\scriptstyle j(C)} & & \downarrow{\scriptstyle j(C'')} & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \end{array}$$

in which both rows are exact. If we assume that $j(D)$ is an isomorphism for all modules D with $l(D) < l(C)$, then we have $j(C)$, $j(C')$, and therefore $j(C'')$, are all isomorphisms.

Retaining the hypothesis that $W(\lambda + \mu) = W(\lambda)$, we use Lemmas 3.2 and 4.1 to prove that $\varphi\psi$ is naturally isomorphic to the identity functor. Hence φ and ψ are isomorphisms from $\mathcal{A}_\lambda$ to $\mathcal{A}_{\lambda+\mu}$ and from $\mathcal{A}_{\lambda+\mu}$ to $\mathcal{A}_\lambda$ respectively. Moreover, φ and ψ are natural inverses.

Proof of Theorem 1.3. Part ①: Given $B \in \mathcal{A}_{\lambda+\mu}$, irreducible, we may assume that $\psi B \in A_\lambda$ is nonzero. Let C be an irreducible quotient of ψB. We have $\operatorname{Hom}(\psi B, C) \cong \operatorname{Hom}(B, \varphi C)$. Let $i: B \to \varphi C$ be the nontrivial map adjoint to the projection from ψB to C. Then i is an injection, since B is irreducible. By the exactness of ψ, we have an injection $\psi i: \psi B \to \psi\varphi C$. By Lemma 3.1, $\psi\varphi C$ is primary. Hence, ψB is primary.

Part ②: Given an irreducible module $A \in \mathcal{A}_\lambda$ and an irreducible subquotient B of φA, ψB is a subquotient of $\psi\varphi A$. The latter is a multiple of A. Hence ψB is a multiple of A. Applying Lemma 3.1 again we conclude that at least one composition factor of φA maps under ψ to a nonzero multiple of A.

Part ③: Suppose an irreducible module $B \in \mathcal{A}_{\lambda+\mu}$ maps under ψ to a nonzero multiple of $A \in \mathcal{A}_\lambda$, A irreducible. We have $\theta(\psi B) = k\theta(A)$, k a positive integer. Hence $\theta(\varphi\psi B) = k\theta(\varphi A)$. Since $\psi B \neq 0$, there exists an injection of B into $\varphi\psi B$; this map is the adjoint to the identity map from ψB to ψB. Thus, $\theta(B)$ occurs in the decomposition of $k\theta(\varphi A)$, and therefore also in the decomposition of $\theta(A)$ into irreducible characters.

5. Discrete series and limits of discrete series

For this section we make the further assumption on G that $\operatorname{rk} K = \operatorname{rk} G$, so that a maximal torus H in K is a Cartan subgroup of G. The differentials of the characters of H form a lattice L in $\mathfrak{h}^*(\mathbf{R})$ (see Section 2 for notation). Let P be a system of positive roots for $\mathfrak{h}$ in $\mathfrak{g}$, and call $\lambda \in L$ P-dominant if $(\lambda | \alpha) \geqq 0$ for all $\alpha \in P$.

If $\lambda \in L$ is regular, there exists a unique irreducible module $D_\lambda \in \mathcal{A}_\lambda$ such that the globalization (see [8], [21]) is square-integrable and such that

$$\nabla(P_\lambda)\theta(D_\lambda) | H' = (-1)^q \textstyle\sum_{s \in W(H:G)} \varepsilon(s)\xi_{s\lambda} \,. \tag{5.1}$$

Here, p_λ is the unique positive root system with respect to which λ is dominant, $\nabla(P_\lambda)$ is the associated Weyl denominator, $q = 1/2 \dim G/K$, $W(H: G) =$ [normalizer of H in G]$/H$, and $\varepsilon(s) = \det s$ (for background see [13]).

Conversely, every square-integrable representation of G has its derived module equivalent to some D_λ. If λ_1 and λ_2 are regular in L, D_{λ_1} is equivalent to D_{λ_2} if and only if there is some $t \in W(H: G)$ such that $t\lambda_1 = \lambda_2$. The collection of modules $\{D_\lambda\}_{\lambda \in L'/W(H:G)}$ is called the discrete series of G [13].

W. Schmid has introduced in [25] a family of invariant eigendistributions $\theta(P, \lambda)$, obtained from the case λ is P-dominant regular by freezing the constants in the local expressions for $\theta(D_\lambda)$ and then allowing λ to vary over all elements of L. Thus, if T is an arbitrary Cartan subgroup of G, ∇_T is the Weyl denominator for T, and $\mathfrak{F}$ is a connected component of $T'(\mathbf{R})$, we can write

$$\nabla_T\theta(D_\lambda)(t) = \sum_{s \in W} c(s, \mathfrak{F}, P)\xi_{s\lambda}(t) \tag{5.2}$$

where $t \in \mathfrak{F}$ and $c(s, \mathfrak{F}, P)$ is a constant which vanishes whenever sL is not contained in L (see Lemmas 5.7 and 5.8 in [12]). We define $\theta(P, \lambda)$ by the same formula when λ is arbitrary in L:

$$\nabla_T\theta(P, \lambda)(t) = \sum_{s \in W} c(s, \mathfrak{F}, P)\xi_{s\lambda}(t) . \tag{5.3}$$

The patching conditions of Hirai [15] ensure that the above formulas in fact define an invariant eigendistribution on G.

The proof of the following lemma is a variant of the proof of formula (3.8):

LEMMA 5.4. *Let λ be P-dominant (but not necessarily regular), and let μ be a P-dominant integral form such that $\lambda + \mu$ is regular. Then*

$$\theta(P, \lambda) = \theta(\psi_\lambda^{\lambda+\mu} D_{\lambda+\mu}) .$$

COROLLARY 5.5. *If λ is also regular, then $\psi_\lambda^{\lambda+\mu} D_{\lambda+\mu} \cong D_\lambda$.*

Proof of Lemma 5.4. We have

$$\theta(F_{-\mu})(t) = \sum_{\nu \in P(F_{-\mu})} \xi_\nu(t) .$$

Let $\{\theta(F_{-\mu})\theta(P, \lambda + \mu)\}_\lambda$ denote that component of $\theta(F_{-\mu})\theta(P, \lambda + \mu)$ consisting of characters of the form $\xi_{s\lambda}$, $s \in W$. Arguing as in the derivation of formula 3.8, we have

$$\begin{aligned} \nabla_T\{\theta(F_{-\mu})\theta(P, \lambda + \mu)\}_\lambda &= \sum_{s \in W} c(s, \mathfrak{F}, P)\xi_{s\lambda}(t) \\ &= \nabla_T\theta(P, \lambda) . \end{aligned}$$

On the other hand, we have

$$\theta(\psi_\lambda^{\lambda+\mu} D_{\lambda+\mu}) = \{\theta(F_{-\mu})\theta(P, \lambda + \mu)\}_\lambda .$$

Combining these formulas, we obtain Lemma 5.4.

The author discovered Lemma 5.4 and Corollary 5.5 in spring of 1975. These results have since been employed in references [14], [18], [26] and [28].

We now state a new result, which follows from the combination of Lemma 5.4 with Theorem 1.3:

COROLLARY 5.6. *If λ is P-dominant and singular, then $\theta(P, \lambda)$ is a (true) character of a primary module.*

Corollary 5.6 is connected with other developments. In January 1975, Schmid informed the author that modulo Blattner's conjecture, one could prove the following:

THEOREM 5.7. *If λ is P-dominant and regular with respect to $W(H: G)$, then $\theta(P, \lambda)$ is a (true) irreducible character.*

After learning about Lemma 5.4, Schmid combined this result with results in [26] to obtain a simpler proof[1] of Theorem 5.7. Then, the author discovered Theorem 1.3, which, in combination with Lemma 5.4, yields Corollary 5.6. The passage from Corollary 5.6 to Theorem 5.7 can be effected via the following elementary argument on K-types:

First, if W is an irreducible K-module and P is a positive root system for $(\mathfrak{h}, \mathfrak{g})$, we denote by Λ the highest weight of W relative to the compact roots in P, and we write $W = W_\Lambda$. If A is a Harish-Chandra module, we let $[A: \Lambda] = \dim \mathrm{Hom}_K(W_\Lambda, A)$.

Definition 5.8. $A \in \mathfrak{A}$ is a lowest K-type module relative to P if there exists a K-type W_Λ with $[A: \Lambda] = 1$, such that if $[A: \Lambda'] \neq 0$, then $\Lambda' = \Lambda + \gamma$, γ a sum (possibly empty) of roots in P.

Definition 5.9. Given P, let ρ_c = one-half the sum of the positive compact roots, and ρ_n = one-half the sum of the positive noncompact roots.

Lemma 5.10 (see proof of Lemma (4.14) of [26]). *Suppose $A \in \mathfrak{A}_\lambda$ is a lowest K-type module, and the lowest highest weight Λ satisfies the condition: $\Lambda - \rho_n$ is dominant with respect to compact roots in P. Then,*

$$\lambda = \Lambda + \rho_c - \rho_n .$$

THEOREM 5.11 (*see* [26]). *For λ as in Theorem 5.7, the module whose character is $\theta(P, \lambda)$ is a lowest K-type module.*

Proof. For very regular λ_1, D_{λ_1} is a lowest K-type module, by an old theorem of Schmid [24]. Given λ above, find μ so that $\lambda + \mu$ is very regular. By Lemma 5.4, $\theta(P, \lambda)$ is the character of $\psi_\lambda^{\lambda+\mu} D_{\lambda+\mu}$. $W_{\Lambda'}$ occurs in $F_{-\mu} \otimes D_{\lambda+\mu}$ only if $\Lambda' = \nu + \Lambda''$, ν a weight $F_{-\mu}$, Λ'' a highest K-weight in $D_{\lambda+\mu}$. Moreover, if Λ_1 is the lowest highest weight in $D_{\lambda+\mu}$, $\Lambda_1 - \mu = \lambda + \rho_n - \rho_c$ is K-dominant and $W_{\Lambda_1 - \mu}$ occurs exactly once in $F_{-\mu} \otimes D_{\lambda+\mu}$.

[1] Lecture at Institute for Advanced Study, February 2, 1976.

Hence, $F_{-\mu} \otimes D_{\lambda+\mu}$ is a lowest K-type module. Moreover, the lowest K-type must lie in $p_\lambda(F_{-\mu} \otimes D_{\lambda+\mu}) = \psi_\lambda^{\lambda+\mu} D_{\lambda+\mu}$. Thus, the latter module is a lowest K-type module.

Theorem 5.7 now follows from Corollary 5.6 and Theorem 5.11, since a primary module having a K-type with multiplicity one must be irreducible. From now on we write $D(P, \lambda)$ for the module with character $\theta(P, \lambda)$, λ as in Theorem 5.7.

If the form λ in Theorem 5.11 is regular, we recover the relevant portions of Theorems 1.3 and 1.4 in [26] (minus the further assumption that G is linear). Corollary 5.5 is a key step in Schmid's proof of the lowest K-type property for the discrete series. Schmid also seems to need his universal lowest K-type modules. It now appears the universal modules are necessary only for the deeper result that D_λ is uniquely characterized by its lowest K-type. We thank N. Wallach for this last observation and for suggesting a variant of the proof of Theorem 5.11.

In [25], Schmid shows that if the discrete series modules have the lowest K-type property, then the D_λ's can be realized as spaces of L^2-harmonic spinors (see Corollary 1.5 of [26]). We now obtain this form of the Langlands conjecture from our Theorem 5.11, without recourse to the universal modules of Schmid, and without the assumption that G is linear.

In his proof [27] of the original Langlands conjecture (realization of discrete series via L^2-Dolbeault cohomology on G/H), Schmid uses a statement (Lemma 4.5) stronger than our Theorem 5.11. However, an examination of the proof of Theorem 4.1 in [27], particularly the proof of Lemma 4.14, shows that after all, only our Theorem 5.11 is needed. Again, we can now drop the assumption that G is linear.

In a sequel to the present paper, we will take up the following topics:

① A closer look at the functors $\varphi_{\lambda+\mu}^\lambda$ and $\psi_\lambda^{\lambda+\mu}$ when $\lambda + \mu$ is regular and λ is singular;

② An application of results from ① to give new information on limits of discrete series, particularly in the case when λ is singular with respect to compact roots;

③ An identification of limits of discrete series with certain of the modules constructed by Enright and Varadarajan (see [28]). Schmid [26] and Wallach [28] have already employed Lemma 5.4 above to identify discrete series modules with some of the Enright-Varadarajan modules.

④ A demonstration of the compatibility between periodicity theory and the theory of parabolic induction of Harish-Chandra modules.

We conclude the present paper by mentioning that we can construct a periodicity theory for the $\mathcal{O}$-category of Bernshtein-Gelfand-Gelfand (see [1]).[2] In fact, periodicity is used implicitly in the proof of the main theorem in [1].

YALE UNIVERSITY NEW HAVEN, CONNECTICUT

BIBLIOGRAPHY

[1] I. N. BERNSHTEIN, I. M. GELFAND and S. I. GELFAND, Structure of representations generated by vectors of highest weight, Func. Anal. and Appl. **5** (1) (1971), 1-8.

[2] N. BOURBAKI, Éléments de Mathématique, Fasc. XXVII, *Algèbre Commutative*, Hermann, 1961.

[3] A. I. FOMIN and N. N. SHAPAVALOV, A property of the characters of real semisimple Lie groups, Func. Anal. and Appl. **8** (3) (1974), 270-271.

[4] I. M. GELFAND and V. A. PONOMAREV, The category of Harish-Chandra modules over the Lie algebra of the Lorentz group, Soviet Math. Doklady **8** (5) (1967), 1065-1068.

[5] ———, The classification of undecomposable infinitesimal representations of the Lorentz group, Soviet Math. Doklady **8** (5) (1967), 1114-1117.

[6] I. M. GELFAND, The cohomology of infinite dimensional Lie algebras; some questions of integral geometry, Actes, Congrès Intern. Math., Tome 1, (1970), 95-111.

[7] HARISH-CHANDRA, Representations of a semisimple Lie group on a Banach space I, Trans. A.M.S. **75** (1953), 185-243.

[8] ———, Representations of semisimple Lie groups II, Trans. A.M.S. **76** (1954), 26-65.

[9] ———, Representations of semisimple Lie groups III., Trans. A.M.S. **76** (1954), 234-253.

[10] ———, The characters of semisimple Lie groups, Trans. A.M.S. **83** (1956), 98-163.

[11] ———, Invariant eigendistributions on a semisimple Lie group, Trans. A.M.S. **119** (1965), 457-508.

[12] ———, Discrete series for semisimple Lie groups I, Acta Math. **113**, (1965), 241-318.

[13] ———, Discrete series for semisimple Lie groups II, Acta Math. **116** (1966), 1-111.

[14] H. HECHT and W. SCHMID, A proof of Blattner's conjecture, Inven. Math. **31** (1976), 129-154.

[15] T. HIRAI, Explicit form of the characters of discrete series representations of semisimple Lie groups, Proc. Symp. Pure Math. XXVI (1973), 281-287.

[16] ———, Some remarks on invariant eigendistributions on semisimple Lie groups, J. Math. Kyoto Univ. **12** (2) (1972), 393-411.

[17] A. U. KLIMYK and V. A. SHIROKOV, On the tensor product of representations of the groups $SO_0(n, 1)$ and $U(n, 1)$, Acad. Sciences Ukrainian S.S.R., Preprint ITP-76-5E (1976).

[18] A. W. KNAPP and N. R. WALLACH, Szegö kernels associated with discrete series, I. A. S. mimeographed notes (1976).

[19] A. W. KNAPP and G. J. ZUCKERMAN, Classification of irreducible tempered representations of semisimple Lie groups, P.N.A.S., **73** (August, 1976), 2178-2180.

[20] B. KOSTANT, On the tensor product of a finite and infinite dimensional representation, J. Func. Analysis **20** (4) (1975), 257-285.

[21] J. LEPOWSKY, Algebraic results on representations of semisimple Lie groups, Trans. A.M.S. **176** (1973), 1-44.

[22] J. LEPOWSKY and N. R. WALLACH, Finite- and infinite-dimensional representations of linear semisimple Lie groups, Trans. A.M.S. **184** (1973), 223-246.

[23] S. MACLANE, *Categories for the Working Mathematician*, Springer-Verlag, 1971.

[2] Quite similar results on category $\mathcal{O}$ have been obtained by J. C. Jantzen: Zur Charakterformel gewisser Darstellungen halbeinfacher Gruppen und Lie-Algebren, Math. Zeitschrift **140** (1974), 127-149.

[24] W. SCHMID, On the realization of the discrete series of a semisimple Lie group, Rice Univ. Studies **56** (1970), 99–108.
[25] ———, On the characters of the discrete series: the Hermitian symmetric case. Inven. Math. **30** (1975), 47–144.
[26] ———, Some properties of square-integrable representations, Ann. of Math. **102** (1975), 535–564.
[27] ———, L^2-cohomology and the discrete series. Ann. of Math. **103** (1976), 375–394.
[28] N. R. WALLACH, On the Enright-Varadarajan modules; a construction of the discrete series, preprint, 1975.

(Received August 3, 1976)