THE THEORY OF CHARACTERS AND THE DISCRETE SERIES FOR SEMISIMPLE LIE GROUPS

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Contents

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1. Introduction

1.1. Summary. The purpose of this survey is to describe, with occasional indications of proofs, some of the main results in harmonic analysis on real semisimple Lie groups. More precisely we shall discuss the theory of characters and some of its important consequences, especially the determination of the discrete series of representations, i.e., the irreducible unitary representations with square integrable matrix coefficients. The central results of this theory were obtained by Harish-Chandra in a series of papers **[4]**, **[5]**, **[6]** in the past two decades, and there are clear indications that the ideas and methods of these articles will continue to play a pivotal and definitive role in Fourier analysis on semisimple Lie groups.

Let G be a connected real semisimple Lie group with finite center. The guiding principle behind Harish-Chandra's approach to harmonic analysis may be formulated as follows: Each Cartan subgroup of G makes a separate contribution to the Plancherel formula on G, the compact Cartan subgroups (when they exist) contribute to the discrete part of the Plancherel formula, and the contribution from an arbitrary Cartan subgroup is determined by the discrete series of a suitable reductive subgroup of G (**[4g]**, **[6i]**). More precisely, let $L = L_I L_R$ be a Cartan subgroup where L_R is a vector group and L_I the maximal compact subgroup of L, and let P be a parabolic subgroup with Langlands decomposition $P = ML_RN$. Then L_I is a (compact) Cartan subgroup of M, and the part of the Plancherel formula due to L is determined by those unitary representations of G that are induced by the representations $man \mapsto \zeta(a)\,\gamma(m)\,(m \in M,\, a \in L_R,\, n \in N)$ of P, where ζ is a unitary character of L_R and γ is a discrete series representation of M (these representations of G are irreducible for almost all ζ and have a finite Jordan decomposition for all ζ). From this point of view the construction of the discrete series is an indispensable prerequisite for carrying out the L^2 Fourier theory on G.

However, the determination of the discrete series turned out to be an extremely difficult problem. The early work of Bargmann [**1**] succeeded in the case $G=SL(2, \boldsymbol{R})$; but its subsequent generalizations aimed at realizing these representations in Hilbert spaces of holomorphic functions on suitable homogeneous spaces of G were only partially successful ([**4d**], [**4e**], [**4f**]; see however [**9**], [**10**], [**11**]). It was only relatively recently that Harish-Chandra succeeded in determining all the characters of the discrete series through a profound study of the differential equations satisfied by them ([**6f**], [**6h**]). Our aim in this article is to sketch an outline of the main features of Harish-Chandra's work leading to this construction. Roughly speaking, there are four parts to this program.

The first deals with the local theory of characters, and more generally, invariant eigendistributions. We shall come to this a little bit later.

In the second part (§3) we assume that G has a compact Cartan subgroup B. The problem here is to construct, corresponding to each regular character ξ of B, an invariant eigendistribution on G which is given on B' by a formula similar to Weyl's formula in the compact case. Note that except in the case of compact G, G will have noncompact Cartan subgroups, and consequently it is not reasonable to expect an invariant eigendistribution to be determined by its values on B'; for example, the characters of the unitary principal series all vanish on B'. However, by imposing an extra *global* condition ((ii) of (3.1.1)) on the distribution, that restricts its behaviour at infinity on G, Harish-Chandra was able to prove the existence of a unique distribution Θ_ξ with the required properties[1] (the condition (ii) of (3.1.1) is nothing more than saying that Θ_ξ is tempered).

In the third part (§6) we examine the behaviour at infinity of the analytic functions which are the Fourier components of the Θ_ξ with respect to a maximal compact subgroup K. Clearly the main question to be settled here is the square integrability of these functions. We describe Harish-Chandra's solution to this problem [**6h**]; it is based on a systematic study of the asymptotic behaviour of K-finite tempered eigenfunctions on G, many ideas of which go back to his papers [**5e**], [**5f**].

In the fourth part (§§5, 7) we consider the closed subspace of $L^2(G)$ spanned by the translates of the Fourier components of the Θ_ξ. This subspace can be shown to be contained in the Hilbert space $^\circ L^2(G)$ spanned by the matrix coefficients of the discrete series of G, and the problem is to show that it is all of $^\circ L^2(G)$, i.e., a completeness question. This was accomplished by Harish-Chandra through the fundamental technique of integrating over the conjugacy classes. The main point here is that this technique reduces the harmonic analysis of the matrix coefficients

[1] In our later notation, $\Theta_\xi=\Theta_\lambda$ when $\xi=\xi_\lambda$.

of the discrete series to questions of Fourier analysis on B, and the fact that we have a distribution Θ_ξ corresponding to each regular character ξ of B is decisive here. This method is similar to that used by Weyl in the compact case. It must be pointed out, however, that this similarity is essentially formal. The conjugacy classes of noncompact G are unbounded, and in order to construct a satisfactory theory of integration over them, it is absolutely essential to overcome convergence problems at infinity (§§4, 5.3–5.5).

Fundamental to all this development and preceding it is the local theory of invariant eigendistributions. We describe briefly in §2 the main results of this aspect of analysis on G. The theorems dealing with this topic were established by Harish-Chandra in two stages – first by reducing them to analogous questions on the Lie algebra, and deducing the latter from general theorems on invariant eigendistributions on semisimple Lie algebras. This survey makes no attempt to discuss either of these in any detail. We wish to point out, however, that the above mentioned reduction to the Lie algebra is much more than a technical device. It actually establishes a remarkable connection between harmonic analysis on G and that on the vector space $\mathfrak{g}$. As a major illustration of this we mention the crucial role played by the theory of Fourier transforms on $\mathfrak{g}$ in the construction of the distributions Θ_ξ (§3.5).

1.2. Notation and preliminaries. We shall work with a real Lie group G, not necessarily connected, with Lie algebra $\mathfrak{g}$; $\mathfrak{g}_c$ is the complexification of $\mathfrak{g}$ and $\mathfrak{G}$, the universal enveloping algebra of $\mathfrak{g}_c$. We write $\mathfrak{Z}$ for the center of $\mathfrak{G}$. The important case is when G is connected, semisimple and has a finite center; but the technical necessities of many proofs make it often convenient to operate in the wider context of groups of class $\mathscr{H}$. We shall say that G is of class $\mathscr{H}$ if it has the following properties: (i) $\mathfrak{g}$ is reductive and $\mathrm{Ad}[G]$ is contained in the connected complex adjoint group of $\mathfrak{g}_c$; (ii) G, and the centralizer of $\mathfrak{g}$ in G, have both finitely many connected components; (iii) if ${}^\circ G = \bigcap_\chi \mathrm{kernel}(\chi)$ where the intersection is over all continuous homomorphisms of G into the positive reals, G splits as ${}^\circ G \times V$ where V is a vector group; and (iv) the analytic subgroup of G defined by $[\mathfrak{g}, \mathfrak{g}]$ is closed. As a rule the structure theory of semisimple groups carries over to this more general case. K will denote a fixed maximal compact subgroup of G, and θ, the corresponding involution of G with $\theta(x) = x^{-1}$, $\forall x \in V$. $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the associated Cartan decomposition of $\mathfrak{g}$, $\mathfrak{k}$ being the Lie algebra of K; $G \simeq K \cdot \exp \mathfrak{p}$ as usual. $\langle \cdot, \cdot \rangle$ denotes a nonsingular G-invariant symmetric bilinear form on $\mathfrak{g} \times \mathfrak{g}$ such that (i) it is the Killing form on $\mathfrak{g}_1 \times \mathfrak{g}_1$ ($\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$), (ii) $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal, and (iii) $X \mapsto \|X\|^2 = -\langle X, \theta X \rangle$ is positive definite on $\mathfrak{g}$, thus converting $\mathfrak{g}$ into a Hilbert space.

For $x \in G$, $D(x)$ is the coefficient of t^l in $\det(\mathrm{Ad}(x) - 1 + t)$ where $l = \mathrm{rank}(G)$. $G' = \{x : x \in G, D(x) \neq 0\}$ is the set of regular points of G. For any Cartan subalgebra

(CSA) $\mathfrak{l}$, the corresponding Cartan subgroup (CSG) is defined as the centralizer L of $\mathfrak{l}$ in G; $L'=L\cap G'$. For any root α of $(\mathfrak{g}, \mathfrak{l})$ we write ξ_α for the corresponding global root on L; α is said to be *real* (*imaginary*) if its values on $\mathfrak{l}$ are all real (imaginary). Any CSG is conjugate to one which is θ-stable; if L above is θ-stable, $L=(L\cap K)\cdot\exp(\mathfrak{l}\cap\mathfrak{p})$. G may not admit compact CSG's, but if it does, they are all conjugate. Suppose $G\subseteq G_c$ where G_c is a complex semisimple group, and $\mathfrak{l}$ is as above. We write L_c for the centralizer of $\mathfrak{l}_c$ in G_c. If $\mu\in\mathfrak{l}_c^*$, we write ξ_μ for the complex character of L_c such that $\xi_\mu(\exp H)=e^{\mu(H)}$ $(H\in\mathfrak{l}_c)$, whenever this exists. If P is a positive system of roots of $(\mathfrak{g}, \mathfrak{l})$, $\delta_P=\frac{1}{2}\sum_{\alpha\in P}\alpha$, and ξ_{δ_P} exists, we write $\Delta_{L,P}=\xi_{-\delta_P}\prod_{\alpha\in P}(\xi_\alpha-1)$. W_L (W_{L_c}) is the normalizer of L in G (L_c in G_c). As usual we say $\lambda\in\mathfrak{l}_c^*$ is *regular* if $\langle\lambda, \alpha\rangle\neq 0$, $\forall\alpha\in P$, *integral* if $2\langle\lambda, \alpha\rangle/\langle\alpha, \alpha\rangle\in \boldsymbol{Z}$, $\forall\alpha\in P$.

$G=KAN$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ are fixed Iwasawa decompositions, with $\mathfrak{a}\subseteq\mathfrak{p}$. $\mathfrak{a}^+$ is the positive chamber in $\mathfrak{a}$, $A^+=\exp\mathfrak{a}^+$, and $\log: A\to\mathfrak{a}$ inverts $\exp:\mathfrak{a}\to A$. $\varrho(H)=\frac{1}{2}\operatorname{tr}(\operatorname{ad}H)_\mathfrak{n}$ $(H\in\mathfrak{a})$. Σ denotes the set of simple roots of $(\mathfrak{g}, \mathfrak{a})$.

A subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is *parabolic* if $\mathfrak{q}_c$ contains a Borel subalgebra of $\mathfrak{g}_c$; the normalizer Q of $\mathfrak{q}$ in G is the corresponding *parabolic subgroup* (*psgrp*). $\mathfrak{q}=\mathfrak{m}_1+\mathfrak{n}_1$, where $\mathfrak{m}_1=\mathfrak{q}\cap\theta(\mathfrak{q})$ is reductive and $\mathfrak{n}_1$ is the nil radical of $\mathfrak{q}\cap[\mathfrak{g}, \mathfrak{g}]$, the sum being direct. If $\mathfrak{c}=\operatorname{center}(\mathfrak{m}_1)\cap\mathfrak{p}$, and $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{c}$ in $\mathfrak{m}_1$, we have the Langlands decomposition $\mathfrak{q}=\mathfrak{m}+\mathfrak{c}+\mathfrak{n}_1$. If $C=\exp\mathfrak{c}$, M_1, the centralizer of $\mathfrak{c}$ in G, $M={}^\circ M_1$ and $N_1=\exp\mathfrak{n}_1$, we have the Langlands decomposition $Q=MCN_1$; moreover $M_1=Q\cap\theta(Q)$ and $\mathfrak{m}$ is the Lie algebra of M. If $\mathfrak{l}$ is a θ-stable CSA we can always choose $\mathfrak{q}$ such that $\mathfrak{c}=\mathfrak{l}\cap\mathfrak{p}$. We denote by d_Q the character $m_1\mapsto|\det\operatorname{Ad}(m_1)_{\mathfrak{n}_1}|^{1/2}$ of M_1.

Let $F\subseteq\Sigma$ and let $\mathfrak{p}_F=\mathfrak{m}_{1F}+\mathfrak{n}_F=\mathfrak{m}_F+\mathfrak{a}_F+\mathfrak{n}_F$ where $\mathfrak{a}_F$ is the null space of F, $\mathfrak{m}_{1F}$ is the centralizer of $\mathfrak{a}_F$ in $\mathfrak{g}$, and $\mathfrak{n}_F$ is the span of the root spaces $\mathfrak{g}_\lambda$ for those positive roots λ of $(\mathfrak{g}, \mathfrak{a})$ which are not in $\boldsymbol{R}\cdot F$. Then $\mathfrak{p}_F$ is parabolic, and the above direct sums are its Langlands decompositions. We call these *standard* and denote their corresponding global counterparts by P_F, M_{1F}, A_F, N_F. Any psgrp is conjugate via K to a unique standard one. $\mathfrak{K}$, $\mathfrak{A}$, $\mathfrak{N}$, $\mathfrak{M}_{1F}$, $\mathfrak{M}_F$, $\mathfrak{A}_F$, $\mathfrak{N}_F$ are the subalgebras (containing 1) of $\mathfrak{G}$ generated respectively by $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$, $\mathfrak{m}_{1F}$, $\mathfrak{m}_F$, $\mathfrak{a}_F$, $\mathfrak{n}_F$. $\mathfrak{Z}_F$ denotes the center of $\mathfrak{M}_{1F}$. We put $d_F=d_{P_F}$.

The elements of $\mathfrak{G}$ act as differential operators in G from both left and right. We use Harish-Chandra's notation and put, for a smooth function f, $a, b\in\mathfrak{G}$, and $x\in G$, $(afb)(x)=f(b; x; a)$; and for $X\in\mathfrak{g}$, $f(X; x)=(d/dt)\,(f(\exp tXx))_{t=0}$, $f(x; X)=(d/dt)\,(f(x\exp tX))_{t=0}$ $(x\in G)$. We denote the adjoint of $a\in\mathfrak{G}$ by $a^\dagger$, so that $\int_G af\cdot g\,dx=\int_G f\cdot a^\dagger g\,dx$ for all $f, g\in C_c^\infty(G)$. $\mathfrak{G}$ also acts on distributions on G; if T is a distribution and $a\in\mathfrak{G}$, $(aT)(f)=T(a^\dagger f)$ $(f\in C_c^\infty(G))$. T is said to be invariant if it is invariant under the inner automorphisms of G; it is said to be an eigendistribution if for some homomorphism $\chi:\mathfrak{Z}\to\boldsymbol{C}$, $zT=\chi(z)\,T$, $\forall z\in\mathfrak{Z}$. If M is any Lie subgroup of G with Lie algebra $\mathfrak{m}$ and if χ is any character of M, then, for $a \in \mathfrak{M}$ (=subalgebra of $\mathfrak{G}$ generated by $(1, \mathfrak{m})$), $\chi \circ a \circ \chi^{-1} \in \mathfrak{M}$; if $\chi = e^{\mu}$ where $\mu \in \mathfrak{m}_c^*$, $a \mapsto \chi \circ a \circ \chi^{-1}$ is the unique automorphism of $\mathfrak{M}$ such that $\chi \circ X \circ \chi^{-1} = X - \mu(X) 1 (X \in \mathfrak{m})$.

Let $\mathfrak{l}$ be a CSA and P a positive system of roots of $(\mathfrak{g}, \mathfrak{l})$. If $\delta = \frac{1}{2} \sum_{\alpha \in P} \alpha$ and X_α are root vectors, then, for any $z \in \mathfrak{Z}$, there exists a unique element $\mu_{\mathfrak{g}/\mathfrak{l}}(z)$ in the subalgebra $\mathfrak{L}$ of $\mathfrak{G}$ generated by $(1, \mathfrak{l})$ such that $z - e^{-\delta} \circ \mu_{\mathfrak{g}/\mathfrak{l}}(z) \circ e^{\delta} \in \sum_{\alpha \in P} \mathfrak{G} X_\alpha$; $\mu_{\mathfrak{g}/\mathfrak{l}}(z)$ is independent of P and $\mu_{\mathfrak{g}/\mathfrak{l}}$ is an isomorphism of $\mathfrak{Z}$ onto the algebra of all elements of $\mathfrak{L}$ that are invariant under the Weyl group of $(\mathfrak{g}_c, \mathfrak{l}_c)$. If $\lambda \in \mathfrak{l}_c^*$, the map

$$\chi_\lambda^{\mathfrak{l}} : z \mapsto \mu_{\mathfrak{g}/\mathfrak{l}}(z)(\lambda) \qquad (z \in \mathfrak{Z})$$

is a homomorphism of $\mathfrak{Z}$ into C; we often write χ_λ for $\chi_\lambda^{\mathfrak{l}}$. $\chi_\lambda = \chi_{\lambda'}$ if and only if λ and λ' are in the same orbit of the Weyl group of $(\mathfrak{g}_c, \mathfrak{l}_c)$; every homomorphism of $\mathfrak{Z}$ into C is of the form χ_λ for some $\lambda \in \mathfrak{l}_c^*$. If $\mathfrak{h}_c$ is another CSA of $\mathfrak{g}_c$ and y is an element of the complex adjoint group of $\mathfrak{g}_c$ such that $\mathfrak{l}_c^y = \mathfrak{h}_c$, then $\chi_\lambda^{\mathfrak{l}} = \chi^{\mathfrak{h}}_{\lambda \circ y^{-1}}$. χ_λ is called *regular* if λ is regular.

Suppose $\mathfrak{m}$ is a subalgebra of $\mathfrak{g}$ which is reductive in $\mathfrak{g}$ and has the same rank as $\mathfrak{g}$. Let $\mathfrak{M}$ be as above, and $\mathfrak{Z}_\mathfrak{m}$ the center of $\mathfrak{M}$. Then there exists a unique injection $\mu_{\mathfrak{g}/\mathfrak{m}}$ of $\mathfrak{Z}$ into $\mathfrak{Z}_\mathfrak{m}$ with the following property: For any CSA $\mathfrak{l} \subseteq \mathfrak{m}$, $\mu_{\mathfrak{g}/\mathfrak{l}} = \mu_{\mathfrak{m}/\mathfrak{l}} \circ \mu_{\mathfrak{g}/\mathfrak{m}}$. $\mathfrak{Z}_\mathfrak{m}$ is a free finite module over $\mu_{\mathfrak{g}/\mathfrak{m}}[\mathfrak{Z}]$, of rank equal to the index of the Weyl group of $(\mathfrak{m}_c, \mathfrak{l}_c)$ in that of $(\mathfrak{g}_c, \mathfrak{l}_c)$. On the other hand, with $\mathfrak{l}$ as above, we have a natural "restriction" isomorphism $p \mapsto p_\mathfrak{l}$, of the algebra $I(\mathfrak{g})$ of G-invariant elements of the symmetric algebra $S(\mathfrak{g}_c)$ onto the algebra of Weyl group invariants of $S(\mathfrak{l}_c)$. So we have a canonical isomorphism $\zeta \mapsto \tilde{\zeta}$ of $\mathfrak{Z}$ onto $I(\mathfrak{g})$ such that $\mu_{\mathfrak{g}/\mathfrak{l}}(\zeta) = (\tilde{\zeta})_\mathfrak{l}$. Suppose $\mathfrak{q} \subseteq \mathfrak{g}$ is a parabolic subalgebra and $\mathfrak{m}_1, \mathfrak{n}_1, Q$ are as defined earlier. Then, for any $z \in \mathfrak{Z}$, $z_1 = d_Q^{-1} \circ \mu_{\mathfrak{g}/\mathfrak{m}_1}(z) \circ d_Q$ is the unique element of the center of the enveloping algebra of $\mathfrak{m}_1$ such that $z - z_1 \in \theta(\mathfrak{n}_1) \mathfrak{G} \mathfrak{n}_1$. For $F \subseteq \Sigma$ we write μ_F for $\mu_{\mathfrak{g}/\mathfrak{m}_{1F}}$.

We assume the reader is familiar with the basic concepts and results of representation theory (cf. [**4a**], [**4b**], [**4c**]). Let $\mathscr{E}(G)$ denote the set of all equivalence classes of irreducible unitary representations of G. If $\pi \in \omega \in \mathscr{E}(G)$, the multiplicities $[\omega : \mathfrak{d}]$ with which the classes $\mathfrak{d} \in \mathscr{E}(K)$ enter in the restriction of π to K are all finite, and, in fact, there is a constant $c > 0$ such that $[\omega : \mathfrak{d}] \leqq c \dim(\mathfrak{d})$, $\forall \omega \in \mathscr{E}(G)$ and $\mathfrak{d} \in \mathscr{E}(K)$. It follows from this that, for $\pi \in \omega \in \mathscr{E}(G)$ and any $f \in C_c^\infty(G)$, the operator $\pi(f) = \int_G f(x) \pi(x) \, dx$ is of trace class, and $\Theta_\omega : f \mapsto \operatorname{tr} \pi(f)$ is an invariant distribution on G that depends only on the class ω. Let χ_ω be the infinitesimal character of ω, so that, for any $z \in \mathfrak{Z}$, $\pi(z) = \chi_\omega(z) \cdot 1$ on the Gårding subspace of π; then $z\Theta_\omega = \chi_\omega(z) \Theta_\omega$, for all $z \in \mathfrak{Z}$. Thus Θ_ω, the *global character* of ω, is an invariant eigendistribution corresponding to the eigenhomomorphism χ_ω. In addition Θ_ω is of the positive definite type, i.e.,

(1) $$\Theta_\omega(f * \tilde{f}) \geqq 0 \qquad (f \in C_c^\infty(G));$$

here $*$ denotes convolution and $\tilde{f}(x)=f(x^{-1})^{\text{conj}}$ $(x\in G)$. Of course the most crucial property of Θ_ω is that it determines ω completely: $\omega_1=\omega_2 \Leftrightarrow \Theta_{\omega_1}=\Theta_{\omega_2}$.

Let $X_1,\dots,X_r$ be an orthonormal basis for $\mathfrak{k}$ and let

(2) $$\Omega=1-(X_1^2+\dots+X_r^2).$$

Ω is independent of the choice of the basis and $\Omega^k=\Omega$, $\forall k\in K$. If $\mathfrak{d}\in\mathscr{E}(K)$, the members of $\mathfrak{d}$ map Ω into a real scalar $c(\mathfrak{d})\geq 1$ and it is known that for some $q\geq 0$, $\dim(\mathfrak{d})=O(c(\mathfrak{d})^q)$ and $\sum_{\mathfrak{d}} c(\mathfrak{d})^{-q}<\infty$. Let $\pi\in\omega\in\mathscr{E}(G)$; $\mathfrak{H}$, the Hilbert space of π; $\mathfrak{H}_{\mathfrak{d}}$ $(\mathfrak{d}\in\mathscr{E}(K))$ the isotypical subspaces of $\mathfrak{H}$ and $E_{\mathfrak{d}}:\mathfrak{H}\to\mathfrak{H}_{\mathfrak{d}}$ the corresponding orthogonal projections. Then, $\forall\varphi\in\mathfrak{H}_{\mathfrak{d}}$, $\varphi'\in\mathfrak{H}$, $f\in C_c^\infty(G)$, and any integer $s\geq 0$,

(3) $$(\pi(f)\,\varphi,\varphi')=c(\mathfrak{d})^{-s}\int_G (\Omega^s f)(x)\,(\pi(x)\,\varphi,\varphi')\,dx,$$

and so $\|\pi(f)\,E_{\mathfrak{d}}\|\leq c(\mathfrak{d})^{-s}\|\Omega^s f\|_1$ ($\|\cdot\|_p$ is the L^p-norm). These estimates, together with the bounds $[\omega:\mathfrak{d}]\leq c\dim(\mathfrak{d})$, easily imply that $\pi(f)$ is of trace class and that for some $C>0$, $q\geq 0$,

(4) $$|\Theta_\omega(f)|\leq C\,\|\Omega^q f\|_1 \qquad (f\in C_c^\infty(G));$$

and further, that if the matrix coefficients of ω are in $L^2(G)$,

(5) $$|\Theta_\omega(f)|\leq C\,\|\Omega^q f\|_2 \qquad (f\in C_c^\infty(G)).$$

The theory of representations and characters is intimately related to the theory of (matrix as well as scalar) spherical functions. To define the latter in sufficient generality we proceed as follows. Let U be a finite-dimensional Hilbert space and $\tau=(\tau_1,\tau_2)$ a unitary double representation of K: This means that τ_1 is a unitary representation and τ_2 a unitary antirepresentation of K in U, such that $\tau_1(k_1)$ and $\tau_2(k_2)$ commute $\forall k_1,k_2\in K$; in view of this we allow τ_1 to act from the left and τ_2 to do so from the right. A function $f:G\to U$ is said to be *τ-spherical* if $f(k_1xk_2)=\tau_1(k_1)\,f(x)\,\tau_2(k_2)$, $\forall x\in G$, $k_1,k_2\in K$. $C^\infty(G:\tau)$ is the space of all τ-spherical functions of class C^∞. Of special interest are those functions in $C^\infty(G:\tau)$ which are $\mathfrak{Z}$-finite. These are all analytic, and they arise in a natural fashion from irreducible representations of G. For example, let $\pi\in\omega\in\mathscr{E}(G)$ and the notation be as in the previous paragraph. Fix $\mathfrak{d}_1,\mathfrak{d}_2\in\mathscr{E}(K)$ and let U be the Hilbert space of linear maps $u:\mathfrak{H}_{\mathfrak{d}_2}\to\mathfrak{H}_{\mathfrak{d}_1}$ with the Hilbert-Schmidt norm. Let $\pi_{\mathfrak{d}}(k)$ $(k\in K,\ \mathfrak{d}\in\mathscr{E}(K))$ be the restriction of $\pi(k)$ to $\mathfrak{H}_{\mathfrak{d}}$ and let $\tau_1(k_1)\,u=\pi_{\mathfrak{d}_1}(k_1)\,u$, $u\tau_2(k_2)=u\pi_{\mathfrak{d}_2}(k_2)$ $(k_1,k_2\in K,$ $u\in U)$. Then $f:x\mapsto E_{\mathfrak{d}_1}\pi(x)\,E_{\mathfrak{d}_2}$ is an element of $C^\infty(G:\tau)$, and $zf=\chi_\omega(z)f$, for all $z\in\mathfrak{Z}$. It is interesting to consider the special case when $\mathfrak{d}_1=\mathfrak{d}_2$ is the trivial class of K; the function f is then spherical in the usual sense $(f(k_1xk_2)=f(x)$, $\forall x\in G$, $k_1,k_2\in K)$ and is an eigenfunction for every element of the centralizer of K in $\mathfrak{G}$.

The importance and usefulness in harmonic analysis of the τ-spherical functions which are eigenfunctions for $\mathfrak{Z}$ lies of course in the fact that they can be studied directly on the group with the help of their differential equations. As an illustration of this remark we mention the following theorem, which can be used to prove the fundamental finite multiplicity theorems of representation theory.

THEOREM 1. *Let τ, U and the other notation be as above. Fix an ideal $\mathfrak{Z}_0$ in $\mathfrak{Z}$ such that $m = \dim(\mathfrak{Z}/\mathfrak{Z}_0) < \infty$. Let $F(\mathfrak{Z}_0:\tau)$ be the space of all $f \in C^\infty(G:\tau)$ such that $zf = 0$, for all $z \in \mathfrak{Z}_0$. Then*

$$\dim(F(\mathfrak{Z}_0:\tau)) \leqq mw \dim(U) \tag{6}$$

where w is the order of a Weyl group of $\mathfrak{g}_c$.

COROLLARY 2. *Let $\omega \in \mathscr{E}(G)$. Then $[\omega:\mathfrak{d}]$ is finite for all $\mathfrak{d} \in \mathscr{E}(K)$ and*

$$[\omega:\mathfrak{d}] \leqq w^{1/2} \dim(\mathfrak{d}) \qquad (\mathfrak{d} \in \mathscr{E}(K)). \tag{7}$$

COROLLARY 3. *Let f be a C^∞ function on G with values in a finite-dimensional vector space V such that* (i) $\dim(\mathfrak{Z}f) < \infty$, (ii) *the left and right translates of f by elements of K span a finite-dimensional space. Then there exist $\alpha, \beta \in C_c^\infty(G)$ invariant under inner automorphisms by elements of K such that $f = \alpha * f * \beta$.*

For a representation-theoretic proof of this see [**6h**, Theorem 1].

2. Local behaviour of invariant eigendistributions

We now take up the description of the local behaviour of invariant eigendistributions on a semisimple group. Throughout this section G is a connected real form of a simply connected complex semisimple Lie group G_c. However, with suitable modifications, the main results may be shown to be valid for all connected reductive groups.

2.1. Formulation of the main theorems. Let V and W be open subsets of G with $W \subseteq V$; Θ, a distribution on V and $\Theta_W = \Theta \mid W$. Θ is said to be $\mathfrak{Z}$-*finite* on W if $\dim(\mathfrak{Z}\Theta_W) < \infty$.

An invariant open set $V \subseteq G$ is said to be *completely invariant* when it has the following property: If $x \in V$ and x_s is the semisimple part in the Jordan decomposition of x, then $x_s \in V$.[2] In Theorems 1–4, V is an arbitrary completely invariant open subset of G.

[2] This definition appears to be slightly weaker than Harish-Chandra's [6e, p. 461], but is actually equivalent to his.

THEOREM 1. *Let Θ be an invariant $\mathfrak{Z}$-finite distribution on V. Then Θ is a locally summable function that is analytic on $V\cap G'$. Let L be a CSG and P any positive system of roots of $(\mathfrak{g}, \mathfrak{l})$. Define $\Phi_{L,P}(a)=\Delta_{L,P}(a)\,\Theta(a)$ $(a\in L'\cap V)$. Then*

$$(1)\qquad \mu_{\mathfrak{g}/\mathfrak{l}}(z)\,\Phi_{L,P}=0,\quad \forall z\in\mathfrak{Z}_\Theta,$$

where $\mathfrak{l}$ is the CSA corresponding to L and $\mathfrak{Z}_\Theta=\{z:z\in\mathfrak{Z},\ z\Theta=0\}$.

Let $\mathfrak{F}$ be the space of all $f\in C^\infty(\mathfrak{l})$ such that $\mu_{\mathfrak{g}/\mathfrak{l}}(z)f=0$, $\forall z\in\mathfrak{Z}_\Theta$. Suppose that $a\in L\cap V$, $\mathfrak{v}$ is a sufficiently small connected neighborhood of 0 in $\mathfrak{l}$, and $\mathfrak{v}^\times=\{H:H\in\mathfrak{v},\ a\exp H\in L'\}$. Then, for each connected component $\mathfrak{v}^+$ of $\mathfrak{v}^\times$, there exists $f_{\mathfrak{v}^+}\in\mathfrak{F}$ such that $\Phi_{L,P}(a\exp H)=f_{\mathfrak{v}^+}(H)$ $(H\in\mathfrak{v}^+)$. It must however be kept in mind that $f_{\mathfrak{v}^+}$ will in general vary with $\mathfrak{v}^+$. It is clearly a very important problem to elucidate the relations that obtain among the $f_{\mathfrak{v}^+}$ on the interfaces between the various $\mathfrak{v}^+$. Theorem 2 deals with this question.

Let L, P be as in Theorem 1. Put $\varpi_{L,P}=\prod_{\alpha\in P}H_\alpha$ and regard $\varpi_{L,P}$ as a differential operator on L. Define

$$(2)\qquad L'(R)=\{a:a\in L,\ \xi_\alpha(a)\neq 1 \text{ for each real root } \alpha\}.$$

THEOREM 2. *Let the notation be as in Theorem 1. Then $\Phi_{L,P}$ extends to an analytic function on $L'(R)\cap V$ while $\varpi_{L,P}\Phi_{L,P}$ extends to a continuous function Ψ_L on $L\cap V$; Ψ_L is independent of the choice of P. If L_1 and L_2 are two CSG's, $\Psi_{L_1}=\Psi_{L_2}$ on $L_1\cap L_2\cap V$.*

THEOREM 3. *Let $\chi:\mathfrak{Z}\to \mathbf{C}$ be a regular homomorphism and let Θ' be an invariant analytic function on $G'\cap V$ such that $z\Theta'=\chi(z)\,\Theta'$, for all $z\in\mathfrak{Z}$. Then Θ' is locally integrable around each point of V.*

Put $\Theta(f)=\int_{G'\cap V}\Theta'(x)f(x)\,dx$ $(f\in C_c^\infty(V))$. Then, in order that the distribution Θ satisfy the differential equations $z\Theta=\chi(z)\,\Theta$ $(z\in\mathfrak{Z})$ on V, it is sufficient that the functions $\Phi_{L,P}$ possess the properties described in Theorem 2.

Let $a\in G$ be a semisimple element. We shall say that a is *semiregular* if the derived algebra of the centralizer of a in $\mathfrak{g}$ has dimension 3.

THEOREM 4. *Let Θ be an invariant locally summable function on V that is analytic and $\mathfrak{Z}$-finite on $G'\cap V$. In order that Θ be $\mathfrak{Z}$-finite on V, it is necessary and sufficient that, for each (semisimple) semiregular $a\in V$, there should exist an open neighborhood $N_a\subseteq V$ of a such that Θ is $\mathfrak{Z}$-finite on N_a.*

Theorems 1 and 2 are due to Harish-Chandra [**6e**], while Theorems 3 and 4 are virtually implicit in his work ([**6c**], [**6e**]; see also [**8**]). Harish-Chandra's method of proving these theorems rests (mainly) on transferring the study of an invariant distribution in a neighborhood of a semisimple point $a\in G$ to the study of an associated invariant distribution in a neighborhood of 0 of the centralizer of a in $\mathfrak{g}$. We now wish to describe this procedure more precisely.

Let $a \in G$ be semisimple and let $\mathfrak{m}_a$ (resp. M_a) be its centralizer in $\mathfrak{g}$ (resp. G). A system $(\mathfrak{u}, V)$ is said to be *adapted to a* if the following conditions are satisfied:

(i) $\mathfrak{u}$ is an M_a-invariant open neighborhood of 0 in $\mathfrak{m}_a$, which is star-like[3] at 0, and which contains the semisimple parts of each of its elements; $V = U^G$ where $U = a \exp \mathfrak{u}$.

(ii) $X \mapsto a \exp X$ is an analytic diffeomorphism of $\mathfrak{u}$ on U.

(iii) For $y \in U$, $D_a(y) = \det(\mathrm{Ad}(y) - 1)_{\mathfrak{g}/\mathfrak{m}_a} \neq 0$.

(iv) If $x \in G$, $X, X' \in \mathfrak{u}$, and $(a \exp X)^x = a \exp X'$, then $x \in M_a$ and $X^x = X'$.

Under these circumstances V can be shown to be open and completely invariant. V and $\mathfrak{u}$ are both connected. We write

$$J_a(X) = \det\left(\frac{\exp(\mathrm{ad}_{\mathfrak{m}_a} X) - 1}{\mathrm{ad}_{\mathfrak{m}_a} X}\right) \qquad (X \in \mathfrak{u}); \tag{3}$$

$J_a(X) > 0$, $\forall X \in \mathfrak{u}$.

Let $I(\mathfrak{m}_a)$ be the algebra of all elements of the symmetric algebra over $(\mathfrak{m}_a)_c$ that are invariant under the adjoint group of $\mathfrak{m}_a$. Let $\zeta \mapsto \tilde{\zeta}$ be the canonical isomorphism (cf. §1.2) of the center of the enveloping algebra of $(\mathfrak{m}_a)_c$ onto $I(\mathfrak{m}_a)$.

THEOREM 5. (i) *Let V be a completely invariant open subset of G, and V_s the set of semisimple points of V. Then, for each $a \in V_s$, there exists $(\mathfrak{u}_a, V_a)$ adapted to a with $V_a \subseteq V$; moreover, for any such choices, $V = \bigcup_{a \in V_s} V_a$.*

(ii) *Let $a \in G$ be semisimple and let $(\mathfrak{u}, V)$ be adapted to a. Then there is a linear isomorphism $\Theta \mapsto \Theta_a$ of the space of invariant distributions on V onto the space of M_a-invariant distributions on $\mathfrak{u}$ with the following properties:* (a) *for any $z \in \mathfrak{Z}$, $(z\Theta)_a = (\mu_{\mathfrak{g}/\mathfrak{m}_a}(z))^{\sim} \Theta_a$,* (b) *$\Theta$ is a locally summable function on V if and only if Θ_a is a locally summable function on $\mathfrak{u}$; moreover, in this case, $\forall X \in \mathfrak{u}$,*

$$\Theta_a(X) = \Theta(a \exp X)\, |D_a(a \exp X)|^{1/2}\, J_a(X)^{1/2}. \tag{4}$$

2.2. Some remarks on the proofs. These theorems are quite difficult to prove and we do not propose to go into their proofs in any detail. We shall restrict ourselves to a few comments on the main lines of argument.

We begin with Theorem 5. Concerning (i), let $a \in G$ be semisimple, and let $V_a(\varepsilon) = (a \exp \mathfrak{u}_a(\varepsilon))^G$ where $\varepsilon > 0$ and

$$\mathfrak{u}_a(\varepsilon) = \{X : X \in \mathfrak{m}_a,\ |\lambda| < \varepsilon, \text{ for all eigenvalues } \lambda \text{ of } \mathrm{ad}\, X\}.$$

One can then prove that $(\mathfrak{u}_a(\varepsilon), V_a(\varepsilon))$ is adapted to a for all sufficiently small

[3] This means that if $X \in \mathfrak{u}$ and $|t| \leq 1$, $tX \in \mathfrak{u}$.

$\varepsilon>0$, and that, given any invariant open set W containing a, $V_a(\varepsilon)\subseteq W$ for some $\varepsilon>0$.

Part (ii) is incomparably more difficult to establish. There are three main stages in its proof. For the first we need the following two lemmas.

LEMMA 1. *Let M (resp. N) be an analytic orientable manifold of dimension m (resp. n), and let ω_M (resp. ω_N) be an analytic m-form (resp. n-form) on M (resp. N) that is everywhere >0. Let ψ be an analytic submersion of M onto N. Then, for each $\alpha\in C_c^\infty(M)$, there exists $f_\alpha\in C_c^\infty(N)$ such that*

$$\int_N gf_\alpha\omega_N=\int_M (g\circ\psi)\,\alpha\omega_M \qquad (g\in C^\infty(N)). \tag{1}$$

The map $\alpha\mapsto f_\alpha$ is linear, maps $C_c^\infty(M)$ onto $C_c^\infty(N)$, and is continuous in the Schwartz topologies; moreover, $\operatorname{supp}(f_\alpha)\subseteq\psi[\operatorname{supp}\alpha]$. *If D_M (resp. D_N) is a C^∞ differential operator on M (resp. N) and if D_M and D_N are ψ-related,[4] then $f_{D^\dagger_M\alpha}=D_N^\dagger f_\alpha$, for all $\alpha\in C_c^\infty(M)$, the adjoints being taken with respect to ω_M and ω_N.*

This is essentially a local result [**6a**, Theorem 1].

For any distribution Θ on N let τ_Θ be the distribution on M such that $\tau_\Theta(\alpha)=\Theta(f_\alpha)$; $\Theta\mapsto\tau_\Theta$ is linear and injective.

LEMMA 2. *For the map $\Theta\mapsto\tau_\Theta$ we have* (i) $\operatorname{supp}(\tau_\Theta)\subseteq\psi^{-1}[\operatorname{supp}\Theta]$, (ii) *if D_M and D_N are as in Lemma 1, $D_M\tau_\Theta=\tau_{D_N\Theta}$,* (iii) *if S is a measurable function on N, S is locally summable on N if and only if $S\circ\psi$ is locally summable on M; in this case, $\tau_S=S\circ\psi$.*

In the context of Theorem 5 we take $M=G\times\mathfrak{u}$, $N=V$, $\psi(x,X)=(a\exp X)^x$, $\omega_M=dxdX$, $\omega_N=dX$. If Θ is invariant we can write $\tau_\Theta=1\otimes\sigma_\Theta$ for a unique distribution σ_Θ on $\mathfrak{u}$, and σ_Θ is M_a-invariant. The map $\Theta\mapsto\sigma_\Theta$ is a linear isomorphism of the space of invariant distributions on V onto the space of M_a-invariant distributions on $\mathfrak{u}$.

The second stage consists in establishing

LEMMA 3. *Given any analytic invariant differential operator E on V, there is an analytic M_a-invariant differential operator $R(E)$ on $\mathfrak{u}$ such that $\sigma_{E\Theta}=R(E)\,\sigma_\Theta$ for all invariant distributions Θ on V.*

For this see [**6e**, §§5–7].

Let $\lambda_a(X)=|D_a(a\exp X)|^{1/2}\,J_a(X)^{1/2}$ $(X\in\mathfrak{u})$. The third step consists in proving that, for any $z\in\mathfrak{Z}$,

[4] This means that $(D_Ng)\circ\psi=D_M(g\circ\psi)$ for all $g\in C^\infty(N)$.

(2) $$R(z)\,S=(\lambda_a^{-1}\circ(\mu_{\mathfrak{g}/\mathfrak{m}_a}(z))^{\sim}\circ\lambda_a)\,S$$

for all M_a-invariant distributions S on $\mathfrak{u}$. Once this is done we obtain at once the relation

(3) $$\sigma_{z\Theta}=(\lambda_a^{-1}\circ(\mu_{\mathfrak{g}/\mathfrak{m}_a}(z))^{\sim}\circ\lambda_a)\,\sigma_\Theta \qquad (z\in\mathfrak{Z})$$

for all invariant distributions Θ on V, so that we may take $\Theta_a=\lambda_a\sigma_\Theta$. The proof of (2) is simple if S is any M_a-invariant C^∞ function on $\mathfrak{u}$; for distributions, (2) follows from the following lemma.

LEMMA 4. *Let F be an analytic M_a-invariant differential operator on $\mathfrak{u}$. Suppose $FS=0$ for every M_a-invariant C^∞ function S on $\mathfrak{u}$. Then $FS=0$ for every M_a-invariant distribution S on $\mathfrak{u}$.*

Theorem 5 follows from these results. However Lemma 4 is difficult to establish and its proof is based on some of the deeper aspects of invariant analysis on reductive Lie algebras [**6d**, Theorems 4 and 5].

Theorem 5 enables one to reduce the proofs of Theorems 1–4 to those of analogous results on reductive Lie algebras. We shall show how this is done for Theorems 1 and 2 while referring the reader to Harish-Chandra's papers ([**6a**], [**6b**], [**6c**], [**6d**]) for the theory of invariant distributions on reductive Lie algebras.

We first consider Theorem 1. By Theorem 5, Θ_a is M_a-invariant and $I(\mathfrak{m}_a)$-finite on $\mathfrak{u}_a$, and hence is a locally summable function on $\mathfrak{u}_a$ by Theorem 1 of [**6d**]. This shows that Θ is a locally summable function around a. As $a\in V_s$ is arbitrary, Θ is a locally summable function on V.

The reduction of Theorem 2 is based on the following lemma; here $a\in G$ is semisimple and $(\mathfrak{u}, V)$ is adapted to a.

LEMMA 5. *Let $\mathfrak{l}\subseteq\mathfrak{m}_a$ be a CSA and let P (resp. P_a) be a positive system of roots of $(\mathfrak{g},\mathfrak{l})$ (resp. $(\mathfrak{m}_a,\mathfrak{l})$). Write $\pi_{P_a}=\prod_{\alpha\in P_a}\alpha$, $\varpi_{P_a}=\prod_{\alpha\in P_a}H_\alpha$, $\varpi_{P/P_a}=\prod_{\alpha\in P/P_a}H_\alpha$. Then there exists a constant $c(\mathfrak{l},P,P_a)\neq 0$ such that, for all $X\in\mathfrak{l}\cap\mathfrak{u}$,*

(4) $$\Delta_{L,P}(a\exp X)=c(\mathfrak{l},P,P_a)\,\pi_{P_a}(X)\,|D_a(a\exp X)|^{1/2}\,J_a(X)^{1/2}.$$

Moreover, if ζ_a is the element of $I(\mathfrak{m}_a)$ whose restriction to $\mathfrak{l}_c$ is $c(\mathfrak{l},P,P_a)\,\varpi_{P/P_a}$, then ζ_a is independent of the choices of $\mathfrak{l}$, P and P_a.

Suppose Θ is an invariant $\mathfrak{Z}$-finite distribution on V. We note that for $X\in\mathfrak{l}\cap\mathfrak{u}$, $a\exp X\in L'$ (resp. $L'(R)$) if and only if no root (resp. no real root) of $(\mathfrak{m}_a,\mathfrak{l})$ vanishes at X. Let $\tilde{\Theta}_a=\zeta_a\Theta_a$. From Lemma 5 we then obtain, for all $X\in\mathfrak{l}\cap\mathfrak{u}$ with $\pi_{P_a}(X)\neq 0$,

$$\Phi_{L,P}(a\exp X)=c(\mathfrak{l},P,P_a)\,\pi_{P_a}(X)\,\Theta_a(X),\qquad (\varpi_{L,P}\Phi_{L,P})(a\exp X)=\tilde{\Theta}_a(X;\varpi_{P_a}\circ\pi_{P_a}).$$

Theorem 2 now follows from Theorems 2 and 3 and Lemma 19 of [**6d**].

2.3. Examples and remarks. Let $\chi: \mathfrak{Z} \to C$ be a homomorphism and let

(1) $\mathfrak{J}(\chi)=\{\Theta: \Theta$ an invariant distribution on $G, z\Theta=\chi(z)\,\Theta, \forall z \in \mathfrak{Z}\}$.

It follows from Theorems 1 and 2 of §2.1 and the theory of differential equations invariant with respect to a finite reflexion group [**13**] that

(2) $$\dim(\mathfrak{J}(\chi)) \leqq Nw$$

where w is the order of a Weyl group of $\mathfrak{g}_c$ and N is the total number of connected components of the various $L_i'(R)$, $L_1, \ldots, L_r$ being a complete system of mutually nonconjugate CSG's of G. In particular there cannot exist more than Nw mutually inequivalent irreducible unitary representations with the same infinitesimal character. In fact we have the following more general result as a direct consequence of (2):[5] Let π be a representation of G in a Banach space V such that (a) all the K-multiplicities of π are finite; (b) π has both global and infinitesimal characters; then there is an integer $r \geqq 1$ and closed π-invariant subspaces $V_0 = V \supseteq V_1 \supseteq \cdots \supseteq V_r = \{0\}$ such that the representations induced in V_i/V_{i+1} are irreducible for all $i=0, 1, \ldots, r-1$.

We should also note that an invariant $\mathfrak{Z}$-finite distribution on G is an analytic function on an open set that is somewhat larger than G' [**4i**, Theorem 6]. Let $'G$ be the set of all semisimple points $a \in G$ whose centralizers in $\mathfrak{g}$ have compact adjoint groups. $'G$ is easily seen to be an invariant open subset of G. Any invariant $\mathfrak{Z}$-finite distribution Θ on a completely invariant open set V is actually an analytic function on $'G \cap V$. In fact, let $a \in {}'G \cap V$ and let $(\mathfrak{u}_a, V_a)$ be adapted to a with $V_a \subseteq V$; then $I(\mathfrak{m}_a)$ contains an elliptic element $\square$, and Θ_a is annihilated on $\mathfrak{u}_a$ by $\square^k + c_1 \square^{k-1} + \cdots + c_k$ for suitable constants $c_1, \ldots, c_k$.

It is an interesting problem to determine $\mathfrak{J}(\chi)$ as explicitly as possible for an arbitrary χ (see [**8**] for many explicit calculations involving $SU(p, q)$). We merely limit ourselves to a consideration of some examples. We write $r(G)$ for the maximum number of mutually nonconjugate CSG's of G.

EXAMPLE 1. $r(G)=1$. Let L be a θ-stable CSG, $\mathfrak{l}$ its Lie algebra. Then $\mathfrak{l}=\mathfrak{c}+\mathfrak{a}$ where $\mathfrak{c}=\mathfrak{l} \cap \mathfrak{k}$, and $\mathfrak{a}=\mathfrak{l} \cap \mathfrak{p}$ is maximal abelian in $\mathfrak{p}$; also $L=CA$ where $C=L \cap K$ and $A=\exp \mathfrak{a}$. For $a \in L$ we write a_I and a_R for the components of a in C and A respectively. Write $\Delta=\Delta_{L,P}$. Let $\mathscr{C}$ be the set of all $\mu \in \mathfrak{l}_c^*$ for which $\exp H \mapsto \exp\{\mu(H)\}$ is well defined on $\exp \mathfrak{c}$.

For any $\mu \in \mathfrak{l}_c^*$ let $\mathfrak{F}(\mu)$ be the space of all analytic functions φ on L such that (i) $\varphi^s = \varepsilon(s)\,\varphi$, $\forall s \in W_L$, and (ii) $v\varphi = v(\mu)\,\varphi$, $\forall v \in \mu_{\mathfrak{g}/\mathfrak{l}}$ [3]. A simple argument shows that $\mathfrak{F}(\mu)=\{0\}$ if $W_{L_c} \cdot \mu \cap \mathscr{C} = \emptyset$. Now $(\mathfrak{g}, \mathfrak{l})$ has no real roots while every semiregular point of G is already in $'G$. From these facts and Theorems 2, 4 and 5 of

[5] This was pointed out to me by Harish-Chandra in 1968.

§2.1 we then obtain the following result: Fix $\lambda\in\mathfrak{l}_c^*$, and for any $\Theta\in\mathfrak{J}(\chi_\lambda)$ let φ_Θ be the analytic function on L such that $\varphi_\Theta(a)=\Delta(a)\,\Theta(a)$ $(a\in L')$; then $\Theta\mapsto\varphi_\Theta$ is a linear isomorphism of $\mathfrak{J}(\chi_\lambda)$ onto $\mathfrak{J}(\lambda)$. In particular, $\mathfrak{J}(\chi_\lambda)=\{0\}$ when $W_{L_c}\cdot\lambda\cap\mathscr{C}=\emptyset$.

For any character (not necessarily unitary) ξ of L let $\mu_\xi\in\mathfrak{l}_c^*$ be defined by $\xi(\exp H)=\exp\{\mu_\xi(H)\}$ $(H\in\mathfrak{l})$. For any $\mu\in\mathfrak{l}_c^*$ let W_μ be the stabilizer of μ in W_{L_c} and let $P(\mu)$ be the space of all polynomials p on $\mathfrak{l}_c^*$ such that (a) p is harmonic[6] with respect to W_μ, and (b) $p(H_1+H_2)=p(H_2)$, $\forall H_1\in\mathfrak{c}$, $H_2\in\mathfrak{a}$. Given any W_L-orbit $\mathfrak{o}$ in $\mathscr{C}$ let $A'(\mathfrak{o})$ be the space spanned by functions of the form $a\mapsto\xi(a)\,p(\log a_R)$ where ξ is a character of L with $\mu_\xi\in\mathfrak{o}$ and $p\in P(\mu_\xi)$; $A'(\mathfrak{o})$ is stable under W_L and we write $A(\mathfrak{o})$ for the subspace of all $\varphi\in A'(\mathfrak{o})$ such that $\varphi^s=\varepsilon(s)\,\varphi$, $\forall s\in W_L$. It is then not difficult to prove that for any $\lambda\in\mathfrak{l}_c^*$, $\mathfrak{F}(\lambda)$ is precisely the direct sum of the spaces $A(\mathfrak{o}_i)$ $(1\leqq i\leqq r)$ where $\mathfrak{o}_1,\dots,\mathfrak{o}_r$ are the distinct W_L-orbits in $W_{L_c}\cdot\lambda\cap\mathscr{C}$.

Let ξ be a character of L. Then the function $\varphi_\xi=\sum_{s\in W_L}\varepsilon(s)\,\xi^s$ lies in $\mathfrak{F}(\mu_\xi)$, and so there exists an invariant eigendistribution Θ_ξ such that $\Delta\cdot(\Theta_\xi\mid L')$ is a nonzero multiple of φ_ξ. It can be verified that for a suitable choice of this constant, Θ_ξ is the character of a principal series representation of G and that all such characters are obtained in this manner (see [**4h**, Theorem 2], which gives the explicit formula for the principal series characters). Thus, if $\xi_1,\dots,\xi_N$ is a maximal set of characters of L with $\mu_{\xi_i}\in W_{L_c}\cdot\lambda$ $(1\leqq i\leqq N)$ such that no two of the ξ_i are conjugate under W_L, the Θ_{ξ_i} $(1\leqq i\leqq N)$ are linearly independent members of $\mathfrak{J}(\chi_\lambda)$ and are precisely all the principal series characters in $\mathfrak{J}(\chi_\lambda)$. It is possible that $N<\dim(\mathfrak{J}(\chi_\lambda))$.

Suppose now that λ is regular. Then $\mathfrak{F}(\lambda)$ is spanned by the functions φ_ξ corresponding to the characters ξ with $\mu_\xi\in W_{L_c}\cdot\lambda\cap\mathscr{C}$, and so $\mathfrak{J}(\chi_\lambda)$ is spanned by the principal series characters that belong to it. If $\mathscr{C}$ is connected and λ has the additional property that $W_{L_c}\cdot\lambda\cap\mathscr{C}$ is a single W_L-orbit, $\dim(\mathfrak{J}(\chi_\lambda))=1$; for a *complex* G, C is connected and the condition on λ implies that the associated principal series representation is irreducible.

EXAMPLE 2. *$r(G)=2$ and the symmetric space G/K has rank* 1. We now have two θ-stable CSG's L and B with Lie algebras $\mathfrak{l}$ and $\mathfrak{b}$. Concerning L we use the same notation as in Example 1. C has at most two connected components and we may assume that $C\subseteq B\subseteq K$. We can select $y\in G_c$ such that y fixes C pointwise and $\mathfrak{l}_c^y=\mathfrak{b}_c$. Let P_L be a positive system of roots of $(\mathfrak{g},\mathfrak{l})$, $\Delta_L=\Delta_{L,P_L}$, and $\Delta_B=\Delta_L\circ y^{-1}$; P_L contains a single real root α. Let $\beta=\alpha\circ y^{-1}$; then $\xi_\alpha=\xi_\beta=1$ on C. The components of $L'(R)$ are of the form $C^+A^\pm$ where C^+ is a component of C and $A^\pm$ are the subsets of A where $\xi_\alpha\gtrless 1$. If $W_{L,I}$ is the group generated by the Weyl reflexions corresponding to the imaginary roots of $(\mathfrak{g},\mathfrak{l})$, then $W_L=W_{L,I}\cup s_\alpha W_{L,I}$; it can further be shown that $y\circ W_{L,I}\circ y^{-1}\subseteq W_B\cup s_\beta W_B$. If C^+ is

[6] This means that $Dp=0$ for all homogeneous W_μ-invariant differential operators (on $\mathfrak{l}_c$) with constant coefficients and positive order.

a connected component of C, there is an $x^+ \in C^+$ that is fixed by W_L (cf. [**6g**, §24] for this structure theory).

Fix a regular $\lambda \in \mathfrak{l}_c^*$ and let $\mu = \lambda \circ y^{-1}$. As in the previous example $\mathfrak{J}(\chi_\lambda) = \{0\}$ if $W_{L_c} \cdot \lambda \cap \mathscr{C} = \emptyset$. Let $\mathfrak{J}^\circ(\chi_\lambda)$ be the subspace of all Θ in $\mathfrak{J}(\chi_\lambda)$ that are 0 on B'. For $\Theta \in \mathfrak{J}^\circ(\chi_\lambda)$ let φ_Θ be the analytic function on CA^+ such that $\varphi_\Theta(a) = \Delta_L(a)\,\Theta(a)$ $(a \in CA^+ \cap L')$. Then Theorems 1–3 of §2.1 imply that $\Theta \mapsto \varphi_\Theta$ is a linear isomorphism of $\mathfrak{J}^\circ(\chi_\lambda)$ onto the subspace $\mathfrak{F}^\circ(\lambda)$ of all analytic functions φ on CA^+ such that (i) $\varphi^s = \varepsilon(s)\,\varphi$, $\forall s \in W_{L,I}$, (ii) $v\varphi = v(\lambda)\,\varphi$, $\forall v \in \mu_{\mathfrak{g}/\mathfrak{l}}[\mathfrak{Z}]$, and (iii) $\varpi_L \varphi$ has boundary values 0 on C. Let $\xi_1, \dots, \xi_N$ be a maximal set of characters of L with $\mu_{\xi_i} \in W_{L_c} \cdot \lambda$, for all i, no two of which are conjugate under W_L; define $\varphi_i = \sum_{s \in W_{L,I}} \varepsilon(s)\,(\xi_i^s + \xi_i^{s_\alpha s})$. Then $\{\varphi_1, \dots, \varphi_N\}$ is a basis for $\mathfrak{F}^\circ(\lambda)$. One may conclude from this that $\mathfrak{J}^\circ(\chi_\lambda)$ is spanned by the principal series characters it contains.

By Theorem 2 of §2.1, $\mathfrak{J}(\chi_\lambda) = \mathfrak{J}^\circ(\chi_\lambda)$ if λ is nonintegral. Now let λ be assumed to be integral. Given $\Theta \in \mathfrak{J}(\chi_\lambda)$, there are constants c_s with $c_{ts} = c_s$ $(s \in W_{B_c}, t \in W_B)$ such that $\Phi_{B, P_B} = \sum_{s \in W_{B_c}} \varepsilon(s)\, c_s \xi_{s\mu}$. We shall now prove the existence of a unique $\Theta_\mu \in \mathfrak{J}(\chi_\lambda)$ such that

$$(3) \qquad \Delta(b)\,\Theta_\mu(b) = \sum_{s \in W_B} \varepsilon(s)\,\xi_{s\mu}(b) \quad (b \in B'), \qquad \sup_{x \in G'} |D(x)|^{1/2} |\Theta_\mu(x)| < \infty .$$

Assume this for a moment; then, if $\{s_1, \dots, s_p\}$ is a system of representatives for $W_B \backslash W_{B_c}$, $\mathfrak{J}(\chi_\lambda)$ is the direct sum of $\mathfrak{J}^\circ(\chi_\lambda)$ and the $\Theta_{s_i\mu}$ $(1 \leqq i \leqq p)$.

For the existence and uniqueness of Θ_μ it is enough to fix a connected component C^+ of C and show that there is exactly one choice of the constants c_s^+ $(s \in W_{B_c})$ for which the function $\varphi = \sum_{s \in W_{B_c}} \varepsilon(s)\, c_s^+ \,\xi_{(s\mu)\circ y}$ on C^+A^+ has the following properties: (i) φ is bounded on C^+A^+, (ii) $\varphi^s = \varepsilon(s)\,\varphi$, $\forall s \in W_{L,I}$, (iii) $\varpi_L \varphi = \sum_{s \in W_B} \xi_{s\mu}$ on C^+. The conditions (i) and (iii) already imply that $c_s^+ = 1$ if $s \in W_B \cup s_\beta W_B$ and $\langle s\mu, \beta \rangle > 0$, while $c_s^+ = 0$ for all other s. The relation $y \circ W_{L,I} \circ y^{-1} \subseteq W_B \cup s_\beta W_B$ shows that for these choices of the c_s^+, (ii) is automatic. In addition to the existence and uniqueness of Θ_μ the above discussion leads to the following formula valid for all $a = b \exp H \in CA^+ \cap L'$ $(b \in C,\ H \in \mathfrak{a}^+)$ [**6g**, §24]:

$$(4) \qquad \Delta_L(a)\,\Theta_\mu(a) = \sum_{s \in W_B} \varepsilon(s)\,\xi_{s\mu}(b) \exp\{-|((s\mu)\circ y)(H)|\}.$$

3. The distributions Θ_λ

3.1. Formulation of the main theorem. Let G be as in §2, and in addition let $\mathrm{rk}(G) = \mathrm{rk}(K)$. We propose to discuss the construction of the invariant eigendistributions on G that will eventually turn out to be the characters of the discrete series of G. Since $\mathrm{rk}(G) = \mathrm{rk}(K)$, G has compact CSG's, all such being connected and mutually conjugate. We fix one of them, say B, $\subseteq K$, and denote its Lie algebra by $\mathfrak{b}$. All roots of $(\mathfrak{g}, \mathfrak{b})$ are imaginary. Let $\mathscr{L}$ be the additive group of all integral elements of $\mathfrak{b}_c^*$, and $\mathscr{L}'$, the subset of all regular elements of $\mathscr{L}$. The basic result of the theory is the following theorem of Harish-Chandra [**6f**, Theorem 3].

THEOREM 1. *Let P be a positive system of roots of $(\mathfrak{g}, \mathfrak{b})$ and let $\Delta = \Delta_{B,P}$. Let $\lambda \in \mathscr{L}'$. Then there exists a unique invariant eigendistribution Θ_λ on G such that*

$$(1)\qquad \begin{aligned} &\text{(i)}\quad \Theta_\lambda(b)\,\Delta(b) = \textstyle\sum_{s\in W_B} \varepsilon(s)\,\xi_\lambda(b) \qquad (b \in B'),\\ &\text{(ii)}\quad \sup_{x\in G'} |D(x)|^{1/2}\,|\Theta_\lambda(x)| < \infty. \end{aligned}$$

In the following sections we shall examine the main steps in Harish-Chandra's proof of this theorem.

3.2. The class $\mathscr{E}$ of invariant open sets. For any $x \in G$ let $\mathfrak{m}_x$ (resp. M_x) be the centralizer of x in $\mathfrak{g}$ (resp. G). x is called *elliptic* if it is in B^G $(=K^G)$, or, equivalently, if x is semisimple and all eigenvalues of $\mathrm{Ad}(x)$ are of modulus unity. Any $x \in G$ can be uniquely written as $a \exp Y$, where a is elliptic, $Y \in \mathfrak{m}_a$ and all eigenvalues of ad Y are real; a is called the *elliptic component* of x. $\mathscr{E}$ is the class of all invariant open subsets V of G with the following property: If $x \in V$ and a is the elliptic component of x, then $a \exp X \in V$ for all $X \in \mathfrak{m}_a$ for which ad X has only real eigenvalues. Members of $\mathscr{E}$ are clearly completely invariant. Given any elliptic $a \in G$ and $\varepsilon > 0$, we put

$$(1)\qquad \begin{aligned} &\mathfrak{g}[\varepsilon] = \{X : X \in \mathfrak{g},\ |\mathrm{Im}\,\lambda| < \varepsilon,\ \forall \text{ eigenvalues } \lambda \text{ of ad}\,X\},\\ &\mathfrak{u}_a[\varepsilon] = \mathfrak{m}_a \cap \mathfrak{g}[\varepsilon], \qquad U_a[\varepsilon] = a \exp \mathfrak{u}_a[\varepsilon], \qquad V_a[\varepsilon] = U_a[\varepsilon]^G. \end{aligned}$$

LEMMA 1. *If $a \in G$ is elliptic, there exists ε_a with $0 < \varepsilon_a < \pi$ such that the system $(\mathfrak{u}_a[\varepsilon], V_a[\varepsilon])$ is adapted to a for any ε with $0 < \varepsilon \leqq \varepsilon_a$.*

LEMMA 2. *Let $a \in B$ and $0 < \varepsilon \leqq \varepsilon_a$. Then*
(i) *$U_a[\varepsilon] \cap B = a \exp(\mathfrak{b} \cap \mathfrak{u}_a[\varepsilon])$, and*
(ii) *$V_a[\varepsilon] \cap B = \bigcup_{s\in W_B} (U_a[\varepsilon] \cap B)^s$.*

LEMMA 3. *$\mathscr{E}$ is closed under finite intersections. If $a \in G$ is elliptic, then $V_a[\varepsilon] \in \mathscr{E}$ for $0 < \varepsilon \leqq \varepsilon_a$. If $V \in \mathscr{E}$ and $a \in V$ is elliptic, there exists ε with $0 < \varepsilon \leqq \varepsilon_a$ such that $V_a[\varepsilon] \subseteq V$.*

LEMMA 4. *Let $V \in \mathscr{E}$. For each $a \in V \cap B$ let δ_a be chosen so that $0 < \delta_a \leqq \varepsilon_a$ and $V_a[\delta_a] \subseteq V$. Then $V = \bigcup_{a\in V\cap B} V_a[\delta_a]$. If $V = G$, there exists a finite subset $F \subseteq B$ such that $G = \bigcup_{a\in F} V_a[\delta_a]$.*

These lemmas are not particularly difficult to prove.

3.3. Reduction to the Lie algebra. Let $\mathfrak{m}$ be a reductive subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. We now wish to formulate two lemmas dealing with certain spaces of distributions on $\mathfrak{m}$ and deduce Theorem 3.1.1 from them.

Let M be the analytic subgroup of G defined by $\mathfrak{m}$. For any CSA $\mathfrak{l}\subseteq\mathfrak{m}$, let $P_{\mathfrak{m},\mathfrak{l}}$ be a positive system of roots of $(\mathfrak{m},\mathfrak{l})$, and let $\pi_{\mathfrak{m},\mathfrak{l}}=\prod_{\alpha\in P_{\mathfrak{m},\mathfrak{l}}}\alpha$, $\varpi_{\mathfrak{m},\mathfrak{l}}=\prod_{\alpha\in P_{\mathfrak{m},\mathfrak{l}}}H_\alpha$. Put $'\mathfrak{l}=\{H: H\in\mathfrak{l},\ \pi_{\mathfrak{m},\mathfrak{l}}(H)\neq 0\}$. Denote by $W_{M,B}$ the subgroup of W_B that comes from M and write

$$p(\mathfrak{m})=\text{index of } W_{M,B} \text{ in } W_{B_c}. \tag{1}$$

Let $\nu\in\mathfrak{b}_c^*$ be regular and take only imaginary values on $\mathfrak{b}$. For any $\varepsilon>0$ let $\mathfrak{u}[\varepsilon]=\mathfrak{m}\cap\mathfrak{g}[\varepsilon]$, and let $\mathfrak{J}_{\nu,\varepsilon}(\mathfrak{m})$ be the vector space of all M-invariant distributions T on $\mathfrak{u}[\varepsilon]$ having the following properties:

$$\begin{aligned}&\text{(i)}\quad (\mu_{\mathfrak{g}/\mathfrak{m}}(z))^{\sim}\,T=\chi_\nu(z)\,T, \text{ for all } z\in\mathfrak{Z}.\\ &\text{(ii)}\quad \text{For each CSA } \mathfrak{l}\subseteq\mathfrak{m},\ \sup_{H\in{}'\mathfrak{l}\cap\mathfrak{u}[\varepsilon]}|\pi_{\mathfrak{m},\mathfrak{l}}(H)\,T(H)|<\infty.\end{aligned}\tag{2}$$

LEMMA 1. *Let* $T\in\mathfrak{J}_{\nu,\varepsilon}(\mathfrak{m})$. *Then* $T=0$ *if and only if* $T(H)=0$, $\forall H\in{}'\mathfrak{b}\cap\mathfrak{u}[\varepsilon]$.

LEMMA 2. $\dim(\mathfrak{J}_{\nu,\varepsilon}(\mathfrak{m}))=p(\mathfrak{m})$.

We shall now indicate how Theorem 3.1.1 may be deduced from these two lemmas. We begin with the uniqueness. It is convenient to prove it in the following form:

LEMMA 3. *Let* $\lambda\in\mathscr{L}'$, $V\in\mathscr{E}$, *and let* Θ *be an invariant distribution on* V *such that* $z\Theta=\chi_\lambda(z)\,\Theta$, $\forall z\in\mathfrak{Z}$. *Suppose that*

$$\begin{aligned}&\text{(i)}\quad \sup_{x\in V\cap G'}|D(x)|^{1/2}|\Theta(x)|<\infty,\\ &\text{(ii)}\quad \Theta(b)=0,\quad \forall b\in V\cap B'.\end{aligned}\tag{3}$$

Then $\Theta=0$.

Let $a\in V\cap B$, and let ε be such that $0<\varepsilon\leqq\varepsilon_a$ and $V_a[\varepsilon]\subseteq V$. It is enough to prove that $\Theta\mid V_a[\varepsilon]=0$ (Lemma 3.2.4), or $\Theta_a=0$ on $\mathfrak{u}_a[\varepsilon]$, in view of Lemma 3.2.1 and Theorem 2.1.5. A simple calculation based on (2.2.4) shows that $\Theta_a\in\mathfrak{J}_{\lambda,\varepsilon}(\mathfrak{m}_a)$. Further, by (ii) of (3), $\Theta_a(H)=0$, $\forall H\in{}'\mathfrak{b}\cap\mathfrak{u}_a[\varepsilon]$. So $\Theta_a=0$ on $\mathfrak{u}_a[\varepsilon]$ by Lemma 1.

To accomplish the construction of Θ_λ we proceed as follows: Let $a\in B$. For each $T\in\mathfrak{J}_{\lambda,\varepsilon}(\mathfrak{m}_a)$, let φ_T be the analytic function on $\mathfrak{b}[\varepsilon]=\mathfrak{b}\cap\mathfrak{u}_a[\varepsilon]$ such that $\varphi_T(H)=\pi_{\mathfrak{m}_a,\mathfrak{b}}(H)\,T(H)$ for all $H\in{}'\mathfrak{b}\cap\mathfrak{u}_a[\varepsilon]$. By Lemma 1, $T\mapsto\varphi_T$ is a linear injection of $\mathfrak{J}_{\lambda,\varepsilon}(\mathfrak{m}_a)$ into the vector space $\mathfrak{U}_{\lambda,\varepsilon}$ of all analytic functions φ on $\mathfrak{b}[\varepsilon]$ such that (i) $\varphi^s=\varepsilon(s)\,\varphi$, $\forall s\in W_{M,B}$, and (ii) $\varphi=\sum_{s\in W_{Bc}}\varepsilon(s)\,c_s e^{s\lambda}$ for suitable constants c_s. But $\dim(\mathfrak{U}_{\lambda,\varepsilon})=p(\mathfrak{m}_a)$, and so, by Lemma 2, $T\mapsto\varphi_T$ is an isomorphism onto $\mathfrak{U}_{\lambda,\varepsilon}$. In particular we can find $T^{(a)}\in\mathfrak{J}_{\lambda,\varepsilon}(\mathfrak{m}_a)$ such that $\varphi_{T^{(a)}}=\sum_{s\in W_B}\varepsilon(s)\,\xi_{s\lambda}(a)\,e^{s\lambda}$.

The expression for $\varphi_{T^{(a)}}$ shows that $T^{(a)} \mid {}'\mathfrak{b} \cap \mathfrak{b}[\varepsilon]$ is invariant under the subgroup of W_B that fixes a. Lemma 3 then implies that $T^{(a)}$ is invariant under M_a. Theorem 2.1.5 and Lemma 2.2.5 now lead to the existence of an invariant eigendistribution $\Theta_\lambda^{(a)}$ on $V_a[\varepsilon]$ such that

$$(4)\quad \begin{aligned} &\text{(i)}\quad \Theta_\lambda^{(a)}(b)\,\Delta(b) = \textstyle\sum_{s\in W_B} \varepsilon(s)\,\xi_{s\lambda}(b),\ b\in a\exp(\mathfrak{b}[\varepsilon]\cap{}'\mathfrak{b}),\\ &\text{(ii)}\quad \sup_{x\in V_a[\varepsilon]\cap G'} |D(x)|^{1/2}\,|\Theta_\lambda^{(a)}(x)| < \infty. \end{aligned}$$

From (ii) of Lemma 3.2.2 it follows that (i) of (4) is valid for all $b\in B'\cap V_a[\varepsilon]$.

For each $a\in B$, select δ_a with $0<\delta_a\leqq\varepsilon_a$ and let $\Theta_\lambda^{(a)}$ be the distribution constructed as above on $V_a[\delta_a]$. If $a, a'\in B$, then $\Theta_\lambda^{(a)}$ and $\Theta_\lambda^{(a')}$ coincide on $B'\cap V_a[\delta_a] \cap V_{a'}[\delta_{a'}]$, and hence on $V_a[\delta_a]\cap V_{a'}[\delta_{a'}]$ by Lemma 3, as the latter belongs to $\mathscr{E}$ in view of Lemma 3.2.3. The existence of Θ_λ on G now follows from Lemma 3.2.4.

3.4. Proof of Lemma 3.3.1. It remains to indicate how the proofs of Lemmas 1 and 2 of §3.3 may be carried out. We consider first Lemma 3.3.1. Let $T\in\mathfrak{J}_{\nu,\varepsilon}(\mathfrak{m})$ be such that $T(H)=0$, for all $H\in{}'\mathfrak{b}\cap\mathfrak{u}[\varepsilon]$. It must be shown that for each CSA $\mathfrak{l}\subseteq\mathfrak{m}$, $T=0$ on ${}'\mathfrak{l}\cap\mathfrak{u}[\varepsilon]$. Let $\mathfrak{l}_I$ (resp. $\mathfrak{l}_R$) be the subspace of all elements of $\mathfrak{l}$ where all the roots of $(\mathfrak{g}, \mathfrak{l})$ take imaginary (resp. real) values. Then the proof uses induction on $\dim(\mathfrak{l}_R)$. We may assume $\dim(\mathfrak{l}_R)>0$; for otherwise, $\mathfrak{l}$ and $\mathfrak{b}$ are conjugate under M and there is nothing to prove. Let $\mu\in\mathfrak{l}_c^*$ be such that $\chi_\mu=\chi_\nu$. Define

$$(1)\qquad {}'\mathfrak{l}(R) = \{H : H\in\mathfrak{l},\ \text{no real root of } (\mathfrak{m}, \mathfrak{l}) \text{ vanishes at } H\}.$$

If Γ is any connected component of ${}'\mathfrak{l}(R)\cap\mathfrak{u}[\varepsilon]$, there are constants $c_s(\Gamma)$ $(s\in W_{L_c})$ such that

$$(2)\qquad \pi_{\mathfrak{m},\mathfrak{l}}(H)\,T(H) = \sum_{s\in W_{L_c}} \varepsilon(s)\,c_s(\Gamma)\exp\{s\mu(H)\} \qquad (H\in{}'\mathfrak{l}\cap\Gamma).$$

One has to prove that $c_s(\Gamma)=0$, for all s, Γ.

LEMMA 1. (i) *There exists $m\in M$ such that $\mathfrak{l}_I^m\subseteq\mathfrak{b}$.* (ii) *Let $\mathfrak{m}_1$ be the centralizer of $\mathfrak{l}_I$ in $\mathfrak{m}$. Then $\mathfrak{l}_I$ is the center of $\mathfrak{m}_1$ and $\mathfrak{l}_R$ is a CSA of $[\mathfrak{m}_1, \mathfrak{m}_1]$.*

For proving (i) let $\mathfrak{c}=\text{center}(\mathfrak{m})$, $\bar{\mathfrak{l}}=\mathfrak{l}\cap[\mathfrak{m},\mathfrak{m}]$ and let $\bar{\mathfrak{l}}_I(\mathfrak{m})$ (resp. $\bar{\mathfrak{l}}_R(\mathfrak{m})$) be the set of points of $\bar{\mathfrak{l}}$ where all roots of $(\mathfrak{m}, \mathfrak{l})$ take only imaginary (resp. real) values. As both $\mathfrak{b}$ and $\mathfrak{l}$ are CSA's of $\mathfrak{m}$, $\mathfrak{c}\subseteq\mathfrak{b}\cap\mathfrak{l}\subseteq\mathfrak{l}_I$; further, $\bar{\mathfrak{l}}_I(\mathfrak{m})\subseteq\mathfrak{l}_I$, $\bar{\mathfrak{l}}_R(\mathfrak{m})\subseteq\mathfrak{l}_R$.[7] A dimension argument then gives $\mathfrak{l}_I=\mathfrak{c}+\bar{\mathfrak{l}}_I(\mathfrak{m})$, $\mathfrak{l}_R=\bar{\mathfrak{l}}_R(\mathfrak{m})$. It is easy to see that for a suitable $m\in M$, $(\bar{\mathfrak{l}}_I(\mathfrak{m}))^m\subseteq\mathfrak{b}\cap[\mathfrak{m},\mathfrak{m}]$; then $\mathfrak{l}_I^m\subseteq\mathfrak{b}$. For proving (ii) we may, in view of (i), assume that $\mathfrak{l}_I\subseteq\mathfrak{b}$. Then both $\mathfrak{b}$ and $\mathfrak{l}$ are CSA's of $\mathfrak{m}_1$, so that $\mathfrak{l}_I\supseteq\mathfrak{b}\cap\mathfrak{l}\supseteq\text{center}(\mathfrak{m}_1)\supseteq\mathfrak{l}_I$. The roots of $(\mathfrak{m}_1, \mathfrak{l})$ are all real and $\mathfrak{l}\cap[\mathfrak{m}_1, \mathfrak{m}_1]$ is spanned

[7] It follows from representation theory that if $\mathfrak{g}_1$ is a *semisimple* subalgebra of $\mathfrak{g}$ and $X\in\mathfrak{g}_1$, then all eigenvalues of $\operatorname{ad}X$ are real (resp. imaginary) if and only if $\operatorname{ad}_{\mathfrak{g}_1}X=\operatorname{ad}X \mid \mathfrak{g}_1$ has this property.

by the H_α's corresponding to them. So $\mathfrak{l}\cap[\mathfrak{m}_1, \mathfrak{m}_1]\subseteq \mathfrak{l}_R$; by dimensionality, $\mathfrak{l}\cap[\mathfrak{m}_1, \mathfrak{m}_1]=\mathfrak{l}_R$.

Lemma 1 leads at once to

LEMMA 2. *Let* $\mathfrak{l}_I[\varepsilon]=\{H: H\in\mathfrak{l}_I, |\beta(H)|<\varepsilon$, *for all roots* β *of* $(\mathfrak{g}, \mathfrak{l})\}$; *and for any simple system S of roots of* $(\mathfrak{m}_1, \mathfrak{l})$, *let* $\mathfrak{l}_R^+(S)=\{H: H\in\mathfrak{l}_R, \beta(H)>0, \forall\beta\in S\}$. *Then the connected components of* $'\mathfrak{l}(R)\cap\mathfrak{u}[\varepsilon]$ *are precisely all the sets of the form* $\mathfrak{l}_I[\varepsilon]+\mathfrak{l}_R^+(S)$ *(as S varies). If* $W_{\mathfrak{m},\mathfrak{l}}(R)$ *is the group generated by the reflexions* s_β $(\beta\in S)$, *then* $W_{\mathfrak{m},\mathfrak{l}}(R)$ *fixes each element of* $\mathfrak{l}_I$ *and acts simply transitively on the collection* $\{\mathfrak{l}_R^+(S)\}$. *If S is fixed and* $\tau\in\mathfrak{l}_c^*$ *is real valued on* $\mathfrak{l}_R$, *there exists* $s\in W_{\mathfrak{m},\mathfrak{l}}(R)$ *such that* $(s\tau)(H)\geq 0$, *for all* $H\in\mathfrak{l}_R^+(S)$.

LEMMA 3. *Let S be as above and* $\beta\in S$. *Define* $\mathfrak{l}_{R,\beta}^+=\{H: H\in\mathfrak{l}_R, \beta(H)=0, \alpha(H)>0, \forall\alpha\in S\backslash\{\beta\}\}$. *Then there is a CSA* $\tilde{\mathfrak{l}}\subseteq\mathfrak{m}$ *such that* $\dim(\tilde{\mathfrak{l}}_R)=\dim(\mathfrak{l}_R)-1$ *and* $\mathfrak{l}_I[\varepsilon]+\mathfrak{l}_{R,\beta}^+\subseteq(\mathfrak{u}[\varepsilon]\cap\tilde{\mathfrak{l}})\cap\mathrm{Cl}(\mathfrak{l}_I[\varepsilon]+\mathfrak{l}_R^+(S))$.

Let $X_{\pm\beta}\in\mathfrak{m}$ be the root vectors corresponding to $\pm\beta$ such that $\beta([X_\beta, X_{-\beta}])=2$. Let $\mathfrak{l}_\beta$ be the null space of β in $\mathfrak{l}$. It suffices to take $\tilde{\mathfrak{l}}=\mathfrak{l}_\beta+\boldsymbol{R}\cdot(X_\beta-X_{-\beta})$.

The proof of Lemma 3.3.1 may now be completed as follows. Take $\Gamma=\mathfrak{l}_I[\varepsilon]+\mathfrak{l}_R^+(S)$ in (2). Then by Lemma 3 and the induction hypothesis, the continuous function on $\mathfrak{u}[\varepsilon]\cap\mathfrak{l}$ that extends $\varpi_{\mathfrak{m},\mathfrak{l}}(\pi_{\mathfrak{m},\mathfrak{l}}(T\mid{}'\mathfrak{l}\cap\mathfrak{u}[\varepsilon]))$ must vanish on $\mathfrak{l}_I[\varepsilon]+\mathfrak{l}_{R,\beta}^+$. So $\sum_{s\in W_{L_c}} c_s(\Gamma)\,e^{s\mu}\equiv 0$ on $\mathfrak{l}_\beta$, leading to the relations

$$(3)\qquad c_s(\Gamma)+c_{s_\beta s}(\Gamma)=0 \qquad (s\in W_{L_c}, \beta\in S).$$

Suppose $c_t(\Gamma)\neq 0$ for some $t\in W_{L_c}$. Select $s'\in W_{\mathfrak{m},\mathfrak{l}}(R)$ such that $(s't\mu)(H)\geq 0$, $\forall H\in\mathfrak{l}_R^+(S)$. Then $c_{s't}(\Gamma)\neq 0$ by (3). On the other hand, it follows easily from the boundedness of $\sum_{s\in W_{L_c}}\varepsilon(s)\,c_s(\Gamma)\,e^{s\mu}$ on Γ that $s\mu(H)\leq 0$, $\forall H\in\mathfrak{l}_R^+(S)$, if $c_s(\Gamma)\neq 0$. So $s't\mu\mid\mathfrak{l}_R=0$. In particular, $\langle s't\mu, \beta\rangle=0$, $\forall\beta\in S$, contradicting the regularity of μ.

3.5. Tempered invariant distributions on $\mathfrak{m}$. Proof of Lemma 3.3.2. We now take up Lemma 3.3.2. The main step in its proof is to show that if an invariant distribution T on $\mathfrak{u}[\varepsilon]$ satisfying (i) of (3.3.2) is tempered,[8] then it also satisfies (ii) therein, and hence belongs to $\mathfrak{J}_{\nu,\varepsilon}(\mathfrak{m})$. It follows from this that $\mathfrak{J}_{\nu,\varepsilon}(\mathfrak{m})$ contains the restrictions to $\mathfrak{u}[\varepsilon]$ of the Fourier transforms of the invariant measures on suitably chosen M-orbits in $\mathfrak{m}$, enabling us to verify that $\dim(\mathfrak{J}_{\nu,\varepsilon}(\mathfrak{m}))=p(\mathfrak{m})$.

LEMMA 1. *Let* Γ_1 *be an M-invariant open subset of* $\mathfrak{m}$; *let* $\mathfrak{l}\subseteq\mathfrak{m}$ *be a CSA and let* $\Gamma=(\Gamma_1\cap{}'\mathfrak{l})^M$. *Suppose f is any M-invariant continuous function on* Γ *such that the distribution defined by f is tempered. Let* $f_\mathfrak{l}=f\mid\Gamma\cap{}'\mathfrak{l}$. *Then, for some integer* $r\geq 0$, *the distribution defined by* $\pi_{\mathfrak{m},\mathfrak{l}}^r f_\mathfrak{l}$ *on* $\Gamma\cap{}'\mathfrak{l}$ *is tempered.*

[8] A distribution defined on an open subset U of a real vector space V is said to be *tempered* if it is continuous in the topology induced by the seminorms $f\mapsto\sup_U|Ef|$ $(f\in C_c^\infty(U))$ where E is a differential operator on V with polynomial coefficients.

For this lemma, see [**6f**, Lemma 17]. The proof of this lemma goes in three stages. In what follows, for any real vector space V, we write $D(V)$ for the algebra of differential operators on V with polynomial coefficients.

Let L be the CSG of G corresponding to $\mathfrak{l}$ and $\bar{M}=M/L\cap M$. For $m\in M$ let $\bar{m}$ be its image in $\bar{M}$ and let $d\bar{m}$ be the invariant measure in $\bar{M}$. For any $g\in C_c^\infty(\Gamma)$ let $\varphi(g:H)=\int_{\bar{M}} g(H^{\bar{m}})\, d\bar{m}\ (H\in\Gamma\cap{}'\mathfrak{l},\ H^{\bar{m}}=H^m)$; $\varphi(g:\cdot)\in C_c^\infty(\Gamma\cap{}'\mathfrak{l})$. Using the well-known formula for integration on $\mathfrak{m}$, we find the following consequence of the tempered nature of f: $\exists D_i\in D(\mathfrak{m})$ such that, for all $g\in C_c^\infty(\Gamma)$,

$$(1)\qquad \left|\int_{\Gamma\cap{}'\mathfrak{l}} \pi_{\mathfrak{m},\mathfrak{l}}(H)^2\varphi(g:H)\, f(H)\, dH\right| \leqq \sum_{1\leqq i\leqq k}\ \sup_{\Gamma}|D_i g|.$$

The second stage consists in "inverting" the map $g\mapsto\varphi(g:\cdot)$. Let $W_{M,L}$ be the normalizer of $\mathfrak{l}$ in M modulo $L\cap M$. Then $W_{M,L}$ is a finite group that acts naturally on $\bar{M}$ and preserves $d\bar{m}$. We select $\bar{\gamma}\in C_c^\infty(\bar{M})$ invariant under this action such that $\int_{\bar{M}}\bar{\gamma}(\bar{m})\, d\bar{m}=1$. Let $\gamma(m)=\bar{\gamma}(\bar{m})\ (m\in M)$ and let C be a compact set in M whose image in $\bar{M}$ contains $\operatorname{supp}\bar{\gamma}$. It is then easy to show that there is a unique $g_\beta\in C_c^\infty(\Gamma)$ such that $g_\beta(H^m)=\bar{\gamma}(\bar{m})\,\bar{\beta}(H)\ (H\in\Gamma\cap{}'\mathfrak{l},\ m\in M,\ \bar{\beta}=\sum_{s\in W_{M,L}}\beta^s)$; we have $\varphi(g_\beta:H)=\bar{\beta}(H)$, $\forall H\in\Gamma\cap{}'\mathfrak{l}$. From (1) we then obtain, for all $\beta\in C_c^\infty(\Gamma\cap{}'\mathfrak{l})$, with w denoting the order of $W_{M,L}$,

$$(2)\qquad \left|\int_{\Gamma\cap{}'\mathfrak{l}} \pi_{\mathfrak{m},\mathfrak{l}}(H)^2\beta(H)\, f(H)\, dH\right| \leqq w^{-1}\sum_{1\leqq i\leqq k}\ \sup_{m\in C,\, H\in\Gamma\cap{}'\mathfrak{l}} |(D_i g_\beta)(H^m)|.$$

For the third step, let $\psi(m:H)=H^m$. Then an elementary analysis of the differential of ψ leads to the following result: If $E\in D(\mathfrak{m})$, there exists an integer $l\geqq 0$, $E_j\in D(\mathfrak{l})$, $\xi_j\in\mathfrak{M}$ (= subalgebra of $\mathfrak{G}$ generated by $(1,\mathfrak{m})$) and analytic functions h_j on M such that, for all $g\in C^\infty(\Gamma)$, $H\in\Gamma\cap{}'\mathfrak{l}$, $m\in M$,

$$(3)\qquad (Ef)(H^m)=\pi_{\mathfrak{m},\mathfrak{l}}(H)^{-l}\sum_{1\leqq j\leqq q} h_j(m)\,(g\circ\psi)(m;\xi:H;E_j)$$

(this follows from Lemmas 3–5 of §3 of [**5a**]).

If we now use (3) in (2) with $E=D_j$, $g=g_\beta$, the required conclusion about $f_{\mathfrak{l}}$ follows without difficulty.

LEMMA 2. *Let T be a tempered invariant distribution on $\mathfrak{u}[\varepsilon]$ satisfying* (i) *of* (3.3.2). *Then $T\in\mathfrak{J}_{v,\varepsilon}(\mathfrak{m})$.*

Let $\mathfrak{l}\subseteq\mathfrak{m}$ be a CSA. Then T is given by (3.4.2) with $\Gamma=\mathfrak{l}_I[\varepsilon]+\mathfrak{l}_R^+(S)$ (cf. §3.4). By Lemma 1, for some integer $r\geqq 0$, the function

$$H\mapsto\pi_{\mathfrak{m},\mathfrak{l}}(H)^r\sum_{s\in W_{L_c}}\varepsilon(s)\, c_s(\Gamma)\, e^{s\mu(H)}$$

defines a tempered distribution on Γ. It is not difficult to deduce from this (cf. [**6f**, Lemma 15]) that $s\mu(H) \leqq 0$, $\forall H \in \mathfrak{l}_R^+(S)$, whenever $c_s(\Gamma) \neq 0$. But then T must satisfy (ii) of (3.3.2).

Let $\mathscr{C}(\mathfrak{m})$ be the Schwartz space of rapidly decreasing functions on $\mathfrak{m}$. For $g \in \mathscr{C}(\mathfrak{m})$, $\hat{g} \in \mathscr{C}(\mathfrak{m})$ is defined by $\hat{g}(X) = \int_{\mathfrak{m}} g(Y)\, e^{i\langle X, Y\rangle}\, dY$. For any tempered distribution T on $\mathfrak{m}$, its Fourier transform $\hat{T}(g) = T(\hat{g})$ $(g \in \mathscr{C}(\mathfrak{m}))$.

Let ν be as in Lemma 3.3.2. Define $H_\nu \in i\mathfrak{b}$ by $\langle H_\nu, H\rangle = \nu(H)$ $(H \in \mathfrak{b})$.

LEMMA 3. *For any $s \in W_{B_c}$, the orbit $(-iH_\nu^s)^M$ is closed and admits an M-invariant measure $\sigma_{\nu,s}$. $\sigma_{\nu,s}$, regarded as a distribution on $\mathfrak{m}$, is tempered, and $\hat{\sigma}_{\nu,s}$ satisfies the differential equations $(\mu_{\mathfrak{g}/\mathfrak{m}}(z))^\sim\, \hat{\sigma}_{\nu,s} = \chi_\nu(z)\, \hat{\sigma}_{\nu,s}$, for all $z \in \mathfrak{Z}$.*

That the orbit X^M is closed and admits an invariant measure for any regular $X \in \mathfrak{m}$ is well known. That it is tempered when regarded as a distribution in $\mathfrak{m}$ is proved in [**5b**] (cf. §5.9). The last assertion is proved by a straightforward calculation.

Let $s_1, \ldots, s_p$ $(p = p(\mathfrak{m}))$ be representatives of $W_{M,B} \backslash W_{B_c}$. By Lemmas 2 and 3, $\hat{\sigma}_{\nu,s_j} \mid \mathfrak{u}[\varepsilon] \in \mathfrak{J}_{\nu,\varepsilon}(\mathfrak{m})$, $1 \leqq j \leqq p$. Since the orbits $(-iH_\nu^{s_j})^M$ $(1 \leqq j \leqq p)$ are disjoint, the σ_{ν,s_j} are linearly independent. So the $\hat{\sigma}_{\nu,s_j}$ are also linearly independent. A simple argument based on Lemma 3.3.1 now implies the linear independence of $\hat{\sigma}_{\nu,s_j} \mid \mathfrak{u}[\varepsilon]$, $1 \leqq j \leqq p$. Thus $\dim(\mathfrak{J}_{\nu,\varepsilon}(\mathfrak{m})) \geqq p(\mathfrak{m})$. On the other hand, if $T \in \mathfrak{J}_{\nu,\varepsilon}(\mathfrak{m})$, there are constants $c_s(T)$ $(s \in W_{B_c}, c_{ts} = c_t, \forall t \in W_{M,B})$ such that $\pi_{\mathfrak{m},\mathfrak{b}}(H)\, T(H) = \sum_{s \in W_{B_c}} \varepsilon(s)\, c_s(T)\, e^{s\nu(H)}$, $\forall H \in {}'\mathfrak{b} \cap \mathfrak{u}[\varepsilon]$. Lemma 3.3.1 then yields the estimate $\dim(\mathfrak{J}_{\nu,\varepsilon}(\mathfrak{m})) \leqq p(\mathfrak{m})$.

REMARK. A simple Fubini argument applied to the integration formula on $\mathfrak{m}$ shows that for almost all ν the invariant measures $\sigma_{\nu,s}$ are tempered $\forall s \in W_{B_c}$. For such ν, the above discussion is certainly valid. The proof of Lemma 3.4.2 for the exceptional (but still regular) ν may then be completed by a limiting process. We may thus avoid using the highly nontrivial Theorem 3 of [**5b**].

3.6. The distribution Θ_λ^*. Let $\lambda \in \mathscr{L}'$. It follows from the preceding theory that there is a unique invariant eigendistribution Θ_λ^* on G such that

$$(1)\qquad \begin{aligned} &\text{(i)}\quad \Theta_\lambda^*(b)\, \Delta(b) = \textstyle\sum_{s \in W_{Bc}} \varepsilon(s)\, \xi_{s\lambda}(b)\ (b \in B'),\\ &\text{(ii)}\quad \sup_{x \in G'} |D(x)|^{1/2}\, |\Theta_\lambda^*(x)| < \infty. \end{aligned}$$

Clearly $z\Theta_\lambda^* = \chi_\lambda(z)\, \Theta_\lambda^*$, $\forall z \in \mathfrak{Z}$. The distribution Θ_λ^* is somewhat less singular than Θ_λ. For instance, on B, Θ_λ^* is a finite Fourier series, and is therefore bounded. In this section we shall formulate analogous results for the other CSG's [**6f**, §24]. These play an important role in invariant analysis on G.

THEOREM 1. *Let L be a CSG and $\mathfrak{l}$ its Lie algebra. Let $W_{\mathfrak{l}}(I)$ be the group* *generated by the Weyl reflexions corresponding to the imaginary roots of* $(\mathfrak{g}, \mathfrak{l})$. *Then* $W_{\mathrm{I}}(I)$ *leaves* L *(and* L'*) invariant and* $\Theta_\lambda^*(a^s)=\Theta_\lambda^*(a)$ $(a\in L', s\in W_{\mathrm{I}}(I))$. *Moreover, if* $\mathfrak{z}$ *is the centralizer of* $\mathfrak{l}_R$ *in* $\mathfrak{g}$, *there is a constant* $C>0$ *such that*

$$|\Theta_\lambda^*(a)| \leq C|\det(\mathrm{Ad}(a)-1)_{\mathfrak{g}/\mathfrak{z}}|^{-1/2} \qquad (a\in L'). \tag{2}$$

We use induction on $\dim(\mathfrak{l}_R)$. We may assume that $\mathfrak{l}$ is θ-stable, $\mathfrak{l}_I\subseteq\mathfrak{b}$, and $\dim(\mathfrak{l}_R)>0$. If $L_I=L\cap K$ and $F=K\cap\exp(-1)^{1/2}\mathfrak{l}_R$, then $L=L_I\exp\mathfrak{l}_R$ and $L_I=F\exp\mathfrak{l}_I$. So $W_{\mathrm{I}}(I)$ leaves fixed each component of L_I and each element of $\exp\mathfrak{l}_R$. Let $\mu\in\mathfrak{l}_c^*$ be such that $\chi_\mu=\chi_\lambda$.

Let L_I^+ be a component of L_I, and let $\mathfrak{m}_1$ be the centralizer of L_I^+ in $\mathfrak{g}$. Clearly $\mathrm{rk}(\mathfrak{g})=\mathrm{rk}(\mathfrak{m}_1)=\mathrm{rk}(\mathfrak{m}_1\cap\mathfrak{k})$. Proceeding as in § 3.4 (and using the same notation) we find that $\mathfrak{l}_R$ is a CSA of $[\mathfrak{m}_1, \mathfrak{m}_1]$, and that the connected components of $L'(R)$ are precisely all sets of the form $L_I^+\exp\mathfrak{l}_R^+(S)$, where L_I^+ is as above and S is a simple system of roots of $(\mathfrak{m}_1, \mathfrak{l})$. Fix L_I^+, S and let c_s be constants such that

$$\Delta_L(a)\,\Theta_\lambda^*(a)=\sum_{s\in W_{L_c}}\varepsilon(s)\,c_s\xi_{s\mu}(a) \qquad (a\in L'\cap L_I^+\exp\mathfrak{l}_R^+(S));$$

here $\Delta_L=\Delta_{L,P_L}$ for some positive system P_L of roots of $(\mathfrak{g}, \mathfrak{l})$.

LEMMA 2. *Fix* $\beta\in S$. *Then we can find a CSA* $\tilde{\mathfrak{l}}$ *with CSG* $\tilde{L}$ *and an element* $y\in G_c$ *having the following properties:* (i) $\tilde{\mathfrak{l}}$ *is* θ*-stable and* $\dim(\tilde{\mathfrak{l}}_R)=\dim(\mathfrak{l}_R)-1$, (ii) $\mathfrak{l}_c^y=\tilde{\mathfrak{l}}_c$, (iii) *y fixes each element of* $L_I^+\exp(\mathfrak{l}_R\cap\mathfrak{l}_\beta)$, (iv) *there is a connected component* $\tilde{L}^+$ *of* $\tilde{L}'(R)$ *such that* $\mathrm{Cl}(\tilde{L}^+)\cap\mathrm{Cl}(L_I^+\exp\mathfrak{l}_R^+(S))$ *contains* $L_I^+\exp\mathfrak{l}_{R,\beta}^+$, *and* (v) $y\circ W_{\mathrm{I}}(I)\circ y^{-1}\subseteq W_{\mathrm{I}}(I)$.

Select root vectors $X_{\pm\beta}\in\mathfrak{m}_1$ corresponding to $\pm\beta$ such that $\beta([X_\beta, X_{-\beta}])=2$ and $X_{-\beta}=-\theta(X_\beta)$ (this is possible). Take $\tilde{\mathfrak{l}}=\mathfrak{l}_\beta+\boldsymbol{R}\cdot(X_\beta-X_{-\beta})$ and $y=\exp(-(-1)^{1/2}\pi(X_\beta+X_{-\beta})/4)$.

Let $\Delta^L=\Delta_L\circ y^{-1}$ and let d_s be constants such that

$$\Delta_L(h)\,\Theta_\lambda^*(h)=\sum_{s\in W_{L_c}}\varepsilon(s)\quad d_s\xi_{s\mu\circ y^{-1}}(h) \qquad (h\in\tilde{L}^+\cap\tilde{L}').$$

From Lemma 2 and Theorem 2.1.2 it follows that $\sum_{s\in W_{L_c}}(c_s-d_s)\,\xi_{s\mu}$ vanishes on $L_I^+\exp(\mathfrak{l}_R\cap\mathfrak{l}_\beta)$. This gives the relations $c_s+c_{s_\beta s}=d_s+d_{s_\beta s}$, $\forall s$. Fix $t\in W_{\mathrm{I}}(I)$ and write $\tilde{c}_s=c_{ts}-c_s$. From Lemma 2 (v), the invariance of $\Theta_\lambda^*\,|\,\tilde{L}^+$ with respect to $W_{\mathrm{I}}(I)$, and the fact that $W_{\mathrm{I}}(I)$ commutes with s_β, we obtain

$$\tilde{c}_s+\tilde{c}_{s_\beta s}=0 \qquad (s\in W_{L_c}, \beta\in S). \tag{3}$$

Further, if $\tilde{c}_s\neq0$, then either c_s or c_{ts} is $\neq0$, and, in either case, $s\mu(H)\leq0$, $\forall H\in\mathfrak{l}_R^+(S)$.

We now argue as in §3.4 to conclude that $\tilde{c}_s = 0$, $\forall s$. Thus $\Theta_\lambda^*(a^s) = \Theta_\lambda^*(a)$ $(a \in L'$, $s \in W_{\mathrm{I}}(I))$.

For proving (2), let $P_{L,I}$ be the set of imaginary roots in P_L. For $a \in L$, write a_I and a_R for its components in L_I and $\exp \mathfrak{l}_R$ $(a = a_I a_R)$. Define, with $\delta = \frac{1}{2}\sum_{\alpha \in P_L} \alpha$,

$$\Delta_L^+(a) = \xi_\delta(a_R) \prod_{\alpha \in P_L \setminus P_{L,I}} (\xi_\alpha(a) - 1), \qquad \Delta_{L,I}(a) = \xi_\delta(a_I) \prod_{\alpha \in P_{L,I}} (\xi_\alpha(a) - 1).$$

It is then an immediate consequence of the invariance under $W_{\mathrm{I}}(I)$ that, for some constant $C > 0$ and all $a \in L' \cap (L_I^+ \exp \mathfrak{l}_R^+(S))$,

$$|\Delta_L^+(a)\, \Theta_\lambda^*(a)| \leqq C \left| \Delta_{L,I}(a)^{-1} \sum_{s \in W_{\mathrm{I}}(I)} \varepsilon(s)\, \xi_{s\mu}(a_I) \right|.$$

It is not difficult to show that the expression on the right is bounded over L_I^+ and that $|\Delta_L^+(a)|^2 = |\det(\mathrm{Ad}(a) - 1)_{\mathfrak{g}/\mathfrak{l}}|$, $\forall a \in L$. The estimate (2) is now immediate.

4. The Schwartz space

Throughout this section G is a group of class $\mathscr{H}$.

4.1. The functions Ξ and σ. For $x \in G$, $\sigma(x) = \sigma(x^{-1})$ is the distance between the cosets K and xK in the Riemannian space G/K. σ is a spherical function, $\sigma(\exp X) = \|X\|$ $(X \in \mathfrak{p})$, and

(1) $$\sigma(xy) \leqq \sigma(x) + \sigma(y) \qquad (x, y \in G).$$

Let π be the unitary representation of G induced by the trivial representation of a minimal psgrp P. The trivial representation of K occurs exactly once in $\pi \mid K$. We define $\Xi(x) = \Xi_G(x) = (\pi(x)\,\psi, \psi)$ $(x \in G)$ where ψ is a unit vector fixed by $\pi[K]$. Ξ does not depend on the choice of P. For $x \in G$ let $x = k \exp H(x)\, n$ where $k \in K$, $H(x) \in \mathfrak{a}$, $n \in N$; $\varrho(X) = \frac{1}{2}\mathrm{tr}(\mathrm{ad}\, X \mid n)$ $(X \in \mathfrak{a})$. Then it follows from the explicit form of π that [**4b**, p. 43]

(2) $$\Xi(x) = \Xi(x^{-1}) = \int_K \exp\{-\varrho(H(xk))\}\, dk \qquad (x \in G).$$

Ξ is an analytic spherical function, $\Xi(1) = 1$, and $0 < \Xi(x) \leqq 1$, for all x. Further, for any $b \in \mathfrak{G}$ let $a_b \in \mathfrak{A}$ be the unique element such that $b - a_b \in \mathfrak{k}\mathfrak{G} + \mathfrak{G}\mathfrak{n}$ [**5e**, Lemma 3]; then, denoting by $\mathfrak{Q}$ the centralizer of K in $\mathfrak{G}$,

(3) $$q\Xi = a_q(-\varrho)\, \Xi \qquad (q \in \mathfrak{Q}).$$

It is well known [7, §3 of Chapter X] that Ξ is uniquely determined by the differential equations (2) and the condition $\Xi(1)=1$. This observation leads to the relation

(4) $$\int_K \Xi(xky)\,dk=\Xi(x)\,\Xi(y) \qquad (x, y\in G).$$

In the following results we collect together a few estimates involving Ξ and σ. Corollary 2 and Theorem 3 make clear the importance of the function Ξ in Fourier analysis.

THEOREM 1. (i) *Given* $a, b\in\mathfrak{G}$, *there exists* $C=C(a, b)>0$ *such that* $|\Xi(a; x; b)|\leqq C\Xi(x)$, $\forall x\in G$.

(ii) *If* E *is any compact subset of* G, *there exists* $C=C(E)>0$ *such that* $\Xi(y_1xy_2)\leqq C\Xi(x)$, $\forall y_1, y_2\in E$, $x\in G$.

(iii) *There exists* $C>0$ *and* $d\geqq 0$ *such that, for all* $h\in A^+$,

(5) $$\exp\{-\varrho(\log h)\}\leqq\Xi(h)\leqq C\exp\{-\varrho(\log h)\}\,(1+\sigma(h))^d.$$

The estimates (i) and (ii) may be derived from the explicit form of the representation π. The proof of (iii) requires a study of the differential equations (3) [**5e**, Theorem 3].

COROLLARY 2. *There exists* $r>0$ *such that* $\Xi^2(1+\sigma)^{-r}\in L^1(G)$.

For some constant $c>0$, $cdx=J(h)\,dk_1dhdk_2$ $(x=k_1hk_2,\ k_1, k_2\in K,\ h\in A^+)$ where

(6) $$J(h)=\prod_{\lambda>0}\{e^{\lambda(\log h)}-e^{-\lambda(\log h)}\}^{m(\lambda)}$$

(the product is over the positive roots of $(\mathfrak{g}, \mathfrak{a})$ and $m(\lambda)$=dimension of the root space of λ; cf. [7, p. 382]). The corollary follows from this and (5).

THEOREM 3.[9] *Fix* p, $1\leqq p\leqq 2$. *Then, given* $a, b\in\mathfrak{G}$, *there exist* $a_i, b_i\in\mathfrak{G}$ $(1\leqq i\leqq m)$ *with the following property: For any* $f\in C^\infty(G)$ *with* $ufv\in L^p(G)$, $\forall u, v\in\mathfrak{G}$,

(7) $$\|\Xi^{-2/p}(afb)\|_\infty\leqq\sum_{1\leqq i\leqq m}\|a_ifb_i\|_p.$$

In view of the closed graph theorem it is enough to prove that $\|\Xi^{-2/p}(ufv)\|_\infty<\infty$, $\forall u, v\in\mathfrak{G}$. Let $g=afb$. By the work of §5.8, there exist $\zeta_j\in\mathfrak{A}$ $(1\leqq j\leqq r)$ such

[9] For $1\leqq p\leqq\infty$, $\|\cdot\|_p$ is the L^p-norm.

that for all $h \in A^+$ and $F \in H^p(A^+, J)$, $|F(h)|^p \leqq \sum_j \|\zeta_j F\|_{p,J}^p$. We apply this estimate to

$$F(h) = \exp\ (2\varrho(\log h)/p)\, g_h(\eta;\, k_1 : k_2;\, \xi)$$

where $\xi, \eta \in \mathfrak{K}$, $g_h(k_1 : k_2) = g(k_1 h k_2)$ $(k_1, k_2 \in K)$, and integrate over $K \times K$. Thus we find that

$$\sup_{h \in A^+} \exp\ (2\varrho(\log h)/p)\ \|\zeta g_h\|_p < \infty, \quad \forall \zeta \in \mathfrak{K} \otimes \mathfrak{K}.$$

By (5) and Sobolev's lemma, $\sup_{h \in A^+} \Xi(h)^{-2/p} \|g_h\|_\infty = \|\Xi^{-2/p} g\|_\infty < \infty$. For details, see [**14b**, §3].

4.2. Definition of $\mathscr{C}(G)$. Following Harish-Chandra [**6h**, §9], we define the Schwartz space $\mathscr{C}(G)$ to be the space of all $f \in C^\infty(G)$ such that for all $m \geqq 0$, $a, b \in \mathfrak{G}$,

(1) $$\mu_{a,b:m}(f) = \|\Xi^{-1}(1+\sigma)^m (afb)\|_\infty < \infty.$$

The seminorms $\mu_{a,b:m}$ convert $\mathscr{C}(G)$ into a Fréchet space (cf. also [**2**] for $G = SL(2, \boldsymbol{R})$). In this definition, $1+\sigma$ may be replaced by any other function having the same growth. It is not difficult to construct C^∞ spherical functions τ, for example, such that (i) $0 < \alpha \leqq (1+\sigma)^{-1}\tau \leqq \beta < \infty$, and (ii) the derivatives $X_1 \cdots X_r \tau Y_1 \cdots Y_s$ $(r+s \geqq 1,\ X_1, Y_j \in \mathfrak{g})$ are all bounded; for instance, if $G = {}^\circ G$, we may take $\tau = 1 - \log \Xi$. That $\mathscr{C}(G)$ is a natural object of study is made clear by the following result which is a consequence of Theorem 4.1.3.

THEOREM 1. *$\mathscr{C}(G)$ is precisely the space of all $f \in C^\infty(G)$ such that $(1+\sigma)^m (afb) \in L^2(G)$, $\forall m \geqq 0$, $a, b \in \mathfrak{G}$, and its topology coincides with the one induced by the seminorms $f \mapsto \|(1+\sigma)^m (afb)\|_2$.*

THEOREM 2. *$C_c^\infty(G)$ is a dense subspace of $\mathscr{C}(G)$, and the natural inclusion map is continuous. $\mathscr{C}(G)$ is a Fréchet algebra under convolution.*

Let χ_t $(t>0)$ be the characteristic function of the set B_t where $\sigma \leqq t$, and $\beta_t = \beta * \chi_t * \beta$, β being spherical and $\in C_c^\infty(G)$. Using the facts that (i) $\sup_{t>0} \|a\beta_t b\|_\infty < \infty$ $(a, b \in \mathfrak{G})$ and (ii) for some $h>0$, $\beta_t = 1$ on B_{t-h}, $\forall t \geqq h$ (cf. (4.1.1)), we find that for any $f \in \mathscr{C}(G)$, $\beta_t f \in C_c^\infty(G)$ and $\beta_t f \to f$ in $\mathscr{C}(G)$ as $t \to +\infty$. For proving the second assertion let $\Xi_s = \Xi(1+\sigma)^{-s}$ $(s>0)$. Then, by (4.1.1),

(2) $$\Xi_s(y^{-1}kx) \leqq \Xi(y^{-1}kx)\,(1+\sigma(y))^s\,(1+\sigma(x))^{-s} \qquad (x, y \in G,\ k \in K).$$

So, if r is as in Corollary 4.1.2 and $s_1 \geqq s+r$, (2) and (4.1.4) give the estimate (with $c = \int_G \Xi^2 (1+\sigma)^{-r}\, dx$)

(3) $$\Xi_{s_1} * \Xi_s(x) \leqq c\,\Xi_s(x) \qquad (x \in G).$$

The estimate (3) shows that, for $f, g\in\mathscr{C}(G)$, $f*g\in\mathscr{C}(G)$, and that the map $f, g\mapsto f*g$ is continuous.

4.3. Tempered distributions on G. A distribution T on G is *tempered* if T has an extension (which is unique by Theorem 4.2.2) to a continuous linear functional on $\mathscr{C}(G)$.

If ψ is a locally summable function on G such that

$$(1)\qquad |\psi(x)|\leqq C\Xi(x)(1+\sigma(x))^p \qquad \text{(for almost all } x\in G)$$

for some $C>0$, $p\geqq 0$, then ψ defines a tempered distribution since $|\int_G \psi f\,dx| \leqq \text{const}\,\mu_{1,1:p+r}(f)$, $\forall f\in C_c^\infty(G)$, r being as in Corollary 4.1.2. We shall discuss in this section two special cases where growth properties analogous to (1) are consequences of the property of being tempered.

The first result is an analogue of the classical result on measures of slow growth [**12**, p. 97]; it is substantially Theorem 4 of [**6h**].

THEOREM 1. *Let $\mathscr{M}$ be a set of nonnegative Borel measures on G. Then the following statements are equivalent:* (i) *there is a continuous seminorm γ on $\mathscr{C}(G)$ such that $|\int f\,dm|\leqq\gamma(f)$, for all $m\in\mathscr{M}$ and all spherical $f\in C_c^\infty(G)$, and* (ii) *there exists $C>0$ and $q\geqq 0$ such that $\int \Xi(1+\sigma)^{-q}\,dm\leqq C$, for all $m\in\mathscr{M}$. In this case, the convergence of the integral is uniform for $m\in\mathscr{M}$.*

Assume (i). Let $G={}^\circ G\cdot V$, $x={}^\circ x{}^1x$ ($x\in G$, ${}^\circ x\in{}^\circ G$, ${}^1x\in V$). For $t_1, t_2>0$, let $C_{t_1,t_2}=\{x:x\in G, \Xi({}^\circ x)\geqq e^{-t_1}, \sigma({}^1x)\leqq t_2\}$. Select spherical $\beta\in C_c^\infty(G)$ with $\beta(x)=\beta(x^{-1})\geqq 0$ for all x and $\int\beta\,dx=1$, and let $f_{t_1,t_2}=\beta*\chi_{t_1,t_2}*\beta$ where χ_{t_1,t_2} is the characteristic function of C_{t_1,t_2}. Let $a>0$ be such that $e^{-a}\Xi(x)\leqq\Xi(y_1xy_2)\leqq e^a\Xi(x)$ and $\sigma({}^1y)\leqq\frac{1}{2}a$, $\forall x\in G$, $y, y_1, y_2\in\operatorname{supp}\beta$. Then $0\leqq f_{t_1,t_2}\leqq 1$, $f_{t_1,t_2}=1$ on C_{t_1-a,t_2-a} and $=0$ outside C_{t_1+a,t_2+a}. Taking $\gamma=\sum_{1\leqq i\leqq n}\mu_{a_i,b_i:s}$ we find, $\forall m\in\mathscr{M}$, $t_1, t_2>0$,

$$\begin{aligned} m(C_{t_1,t_2}) &\leqq \sum \mu_{a_i,b_i:s}(f_{t_1+a,t_2+a}) \\ &\leqq \sum \|a_i\beta\|_1\,\|\beta b_i\|_1 \sup\{\Xi(x)^{-1}(1+\sigma(x))^s : x\in C_{t_1+2a,t_2+2a}\}. \end{aligned}$$

So there exist $B>0$ and $l\geqq 0$ such that, for all $m\in\mathscr{M}$, $t_1, t_2>0$,

$$(2)\qquad m(C_{t_1,t_2})\leqq Be^{t_1}(1+t_1)^l(1+t_2)^l.$$

The estimate (2) implies (ii). Actually (2) is equivalent to (ii).

Our second result deals with the $\mathfrak{Z}$-finite K-finite functions. It is essentially Theorem 9 of [**6h**].

THEOREM 2. *Let $\varphi\in C^\infty(G)$ be $\mathfrak{Z}$-finite and K-finite. If the distribution defined by φ is tempered, then φ satisfies* (1) *for some $C>0$, $p\geqq 0$.*

Write $T_\varphi(f)=\int \varphi f\,dx$ $(f\in C_c^\infty(G))$. Then there exist $a_i, b_i\in\mathfrak{G}$ and $p\geqq 0$ such that, for all $f\in C_c^\infty(G)$),

$$|T_\varphi(f)|\geqq \sum_{1\leqq i\leqq n} \mu_{a_i, b_i:p}(f). \tag{3}$$

Now, by Corollary 1.2.3, there exist $\beta_1, \beta_2\in C_c^\infty(G)$ such that $\varphi=\beta_1*\varphi*\beta_2$ so that $T_\varphi(f)=T_\varphi(\beta_1'*f*\beta_2')$ $(\beta_i'(x)=\beta_i(x^{-1}))$. Transferring the differentiations in (3) to the β_i' we find that, for some $C>0$, $|T_\varphi(f)|\leqq C\mu_{1,1:p}(f)$, $\forall f\in C_c^\infty(G)$. An elementary measure-theoretic argument then gives $|\varphi|\ \Xi(1+\sigma)^{-p}\in L^1(G)$. Now, if $a, b\in\mathfrak{G}$, $a\varphi b$ is both $\mathfrak{Z}$-finite and K-finite and $T_{a\varphi b}$ satisfies an estimate of the form (3) with the *same* p. Hence $|a\varphi b|\ \Xi(1+\sigma)^{-p}\in L^1(G)$. Replacing σ by a function τ as described in §4.2 we find $a(\Xi(1+\tau)^{-p}\varphi)\,b\in L^1(G)$, $\forall a, b\in\mathfrak{G}$. Theorem 4.1.3 now implies that $\|\Xi^{-1}(1+\tau)^{-p}a\varphi b\|_\infty<\infty$ for all $a, b\in\mathfrak{G}$.

As one may infer from Harish-Chandra's work **[6i]**, only those irreducible unitary representations (the so-called tempered representations) whose characters and matrix coefficients are tempered distributions play a role in the L^2 Fourier theory on G, and therein lies the real importance of the tempered distributions. For instance, the representations of the discrete series are tempered, as one may conclude from the L^2 estimates (1.2.5), as are the representations that are associated with the various CSG's (cf. §1.1). If π is an irreducible unitary representation whose character Θ_π is tempered, the matrix coefficients of π defined by K-finite vectors are of the form $a(\Theta_\pi * f)\,b$ where $a, b\in\mathfrak{G}$, f is a matrix coefficient of K, and the convolution is over K; they are therefore tempered. The converse is also true, though less trivial to establish: If the matrix coefficients defined by K-finite vectors of an irreducible unitary π are tempered, then Θ_π is tempered.

It is an important problem to determine the conditions for an invariant $\mathfrak{Z}$-finite distribution to be tempered. We shall come to this later.

5. Invariant analysis on G

In this section we shall assume that G is of class $\mathscr{H}$ and discuss some aspects of the theory of integration over the conjugacy classes of G. This is one of the most important techniques of harmonic analysis; it enables us to reduce many problems on G to (presumably easier) questions on the Cartan subgroups of G **[6h]**.

We set up the map $f\mapsto {}'F_{f,L}$ for a CSG L, and formulate the main results in §5.1. In §5.2 we examine how these can be reduced to the case of compact L. The key estimate used in the proofs is discussed in §5.4. These are then applied in §§5.5 and 5.7 to study various questions on harmonic analysis. There are two appendices: The first (§5.8) deals with some estimates of a classical nature; the second (§5.9) examines some aspects of the theory of tempered invariant eigendistributions on a real semisimple Lie algebra.

5.1. The map $f\mapsto {}'F_f$. Let $\mathfrak{l}$ be a θ-stable CSA, $\mathfrak{l}_R=\mathfrak{l}\cap\mathfrak{p}$, $\mathfrak{l}_I=\mathfrak{l}\cap\mathfrak{k}$, and $\mathfrak{L}$, the subalgebra of $\mathfrak{G}$ generated by $(1, \mathfrak{l})$; L is the corresponding CSG, $L_I=L\cap K$, $L_R=\exp\mathfrak{l}_R$. M_1 (resp. $\mathfrak{m}_1$) is the centralizer of $\mathfrak{l}_R$ in G (resp. $\mathfrak{g}$); $M={}^\circ M_1$, and $\mathfrak{m}$, the Lie algebra of M. $D_\mathfrak{l}$ is the invariant function on G vanishing outside $(L')^G$ such that $D_\mathfrak{l}(b)=\det(1-\mathrm{Ad}(b))_{\mathfrak{g}/\mathfrak{m}_1}$ $(b\in L')$; if L is compact, $D_\mathfrak{l}$ is the characteristic function of the regular elliptic set $(L')^G$, and is denoted also by φ_L. P_I is a positive system of roots of $(\mathfrak{m}_1, \mathfrak{l})$; ${}'\Delta_I=\prod_{\alpha\in P_I}(\xi_\alpha-1)$; $\delta_I=\frac{1}{2}\sum_{\alpha\in P_I}\alpha$. $\bar{G}=G/L_R$, $x\mapsto\bar{x}$ is the natural map of G on $\bar{G}$, and $d\bar{x}$ is an invariant measure on $\bar{G}$; if $b\in L$, $x\in G$, we write $b^{\bar{x}}$ for b^x. $L'(I)$ is the set of all $b\in L$ such that $\xi_\beta(b)\neq 1$ for any singular[10] $\beta\in P_I$. For any open set $U\subseteq L$, let $\mathscr{C}(U)$ be the Schwartz space of U. For any continuous function f on G we put

$$(1)\qquad {}'F_f(b)={}'F_{f,L}(b)={}'\Delta_I(b)\,|D_\mathfrak{l}(b)|^{1/2}\int_{\bar{G}} f(b^{\bar{x}})\,dx \qquad (b\in L')$$

whenever this integral is absolutely convergent for all $b\in L'$. Put

$$(2)\qquad {}'\zeta=e^{\delta_I}\circ\zeta\circ e^{-\delta_I} \qquad (\zeta\in\mathfrak{L}).$$

THEOREM 1. *For each $f\in\mathscr{C}(G)$, ${}'F_f$ is well defined and lies in $\mathscr{C}(L')$. The map $f\mapsto{}'F_f$ is continuous from $\mathscr{C}(G)$ to $\mathscr{C}(L')$. Moreover, for all $z\in\mathfrak{Z}$, $f\in\mathscr{C}(G)$,*

$$(3)\qquad {}'F_{zf}={}'\mu_{\mathfrak{g}/\mathfrak{l}}(z)\,{}'F_f.$$

THEOREM 2. *Fix $f\in\mathscr{C}(G)$ and $b\in L$. (i) If $b\in L'(I)$, then ${}'F_f$ extends as a C^∞ function around b. (ii) Let $b\notin L'(I)$ and let $S_I(b)$ be the set of all singular $\beta\in P_I$ with $\xi_\beta(b)=1$. Then, for any $\zeta\in\mathfrak{L}$ for which $\zeta^{s_\beta}=-\zeta$, $\forall\beta\in S_I(b)$, ${}'\zeta\,{}'F_f$ extends as a continuous function around b. In particular, if $\varpi_I=\prod_{\beta\in P_I}H_\beta$, ${}'\varpi_I{}'F_f$ extends to a continuous function on L.*

Let $\beta\in P_I$ be singular, $L_\beta=\{b:b\in L,\ \xi_\beta(b)=1\}$, and let L'_β be the set of all $b\in L_\beta$ such that $\xi_{\pm\beta}$ are the only global roots that are equal to 1 at b. Let $\mathfrak{z}$ be the centralizer of L_β in $\mathfrak{g}$ and select a CSA $\mathfrak{l}_1\subseteq\mathfrak{z}$ that is not conjugate to $\mathfrak{l}$ in G; let L_1 be the corresponding CSG. Note that $b\in L'_1(I)$ and so, for any $f\in\mathscr{C}(G)$, ${}'F_{f,L_1}$ extends (by Theorem 2) to a C^∞ function in a neighborhood of b in L_1. For any function $g\in\mathscr{C}(L'(I))$ and any $b\in L'_\beta$ we write

$$g(b\pm)=\lim_{t\to 0\pm} g(b\exp(-1)^{1/2}tH_\beta).$$

Let y be an element in the adjoint group of $\mathfrak{z}_c$ such that $\mathfrak{l}_c^y=(\mathfrak{l}_1)_c$.

[10] Given $\beta\in P$ which is either real or imaginary, let $\mathfrak{z}=\mathfrak{g}\cap(C\cdot H_\beta+C\cdot X_\beta+C\cdot X_{-\beta}+C\cdot X_{-\beta})$. β is called singular if $\mathfrak{z}$ is not of compact type.

THEOREM 3. *Let the notation be as above. Then there is an automorphism $\zeta \to \mathring{\zeta}$ of $\mathfrak{L}$ and a C^∞ function c on L'_β such that, for all $\sigma \in \mathfrak{L}$, $b \in L'_\beta$,*

$$(4) \qquad (\zeta\, 'F_{f,L})(b+) - (\zeta\, 'F_{f,L})(b-) = c(b)\,(\mathring{\zeta}^\nu\, 'F_{f,L_1})(b).$$

Suppose G is as in §3. The above results then suggest a simple modification of the definition of $'F_f$ to simplify some of its formal properties. Let L be a CSG; $\Delta = \Delta_{L,P}(P = P_L)$ is a positive system of roots of $(\mathfrak{g}, \mathfrak{l})$. Let

$$\varepsilon_R(b) = \operatorname{sign} \prod_{\alpha \in P,\, \alpha \text{ real}} (\xi_\alpha(b) - 1) \qquad (b \in L').$$

Then we define, for all $f \in \mathscr{C}(G)$,

$$(5) \qquad F_f(b) = F_{f,L}(b) = \varepsilon_R(b)\, \Delta(b) \int_{\bar{G}} f(b^{\bar{x}})\, d\bar{x} \qquad (b \in L').$$

Changing P to another positive system P' results in merely multiplying the RHS of (5) by a constant $C(P, P') = \pm 1$. Let us now choose P so that it contains the complex conjugate of each of its nonimaginary roots, and let P_I be the set of imaginary roots in P. Then a simple calculation shows that

$$(6) \qquad F_f(b) = \xi_{-\delta}(b_I)' F_f(b) \qquad (b \in L',\ f \in \mathscr{C}(G))$$

where $\delta = \frac{1}{2} \sum_{\alpha \in P} \alpha$. So

$$(7) \qquad F_{zf} = \mu_{\mathfrak{g}/\mathfrak{l}}(z)\, F_f \qquad (f \in \mathscr{C}(G),\ z \in \mathfrak{Z}).$$

Moreover, if ζ is as in Theorem 2, ζF_f extends continuously around b. In particular, $\varpi_I F_f$ extends to a continuous function on L. Finally, Theorem 3 may be reformulated in the following manner: There is a nowhere vanishing locally constant function c on L'_β such that, for all $\zeta \in \mathfrak{L}$, $b \in L'_\beta$,

$$(8) \qquad (\zeta F_{f,L})(b+) - (\zeta F_{f,L})(b-) = c(b)\,(\zeta^\nu F_{f,L_1})(b).$$

5.2. Reduction to a compact CSG. The first step in proving these theorems is to come down to the case when L is compact. Let $Q = MCN_1$ be a psgrp with $C \subseteq L_R$. For any continuous function f on G we write

$$(1) \qquad f_Q(m_1) = d_Q(m_1) \int_{N_1} f(m_1 n)\, dn \qquad (m_1 \in M_1 = MC)$$

whenever this integral is absolutely convergent for all m_1 (certainly for $f \in C_c(G)$). If $f \in C_c^\infty(G)$, then $f_Q \in C_c^\infty(M_1)$, and for all $z \in \mathfrak{Z}$, $a, b \in \mathfrak{m}_1$,

(2) $$bf_Q a = ('bfa')_Q \qquad (a' = d_Q^{-1} \circ a \circ d_Q, \; 'b = d_Q \circ b \circ d_Q^{-1})$$
$$(zf)_Q = \mu_{\mathfrak{g}/\mathfrak{m}_1}(z) f_Q.$$

If $f \in C_c(G)$ is spherical, $f_Q \in C_c(M_1)$ and is spherical on M_1; moreover,

(3) $$\int_{M_1} \Xi_{M_1} f_Q \, dm_1 = \int_G \Xi f \, dx$$

with dx and dm_1 suitably normalized (independently of f). These results are not difficult to establish.

LEMMA 1. *There exist $q \geqq 0$, and for each $l \geqq 0$ a constant $C_l > 0$, such that for all $m_1 \in M_1$,*

(4) $$d_Q(m_1) \int_{N_1} \Xi(m_1 n)(1 + \sigma(m_1 n))^{-(q+l)} \, dn \leqq C_l \Xi_{M_1}(m_1)(1 + \sigma(m_1))^{-l},$$

the integrals converging uniformly when m_1 varies over compact subsets of M_1.

First assume $l = 0$. Let r be such that $c = \int_G \Xi^2 (1 + \sigma)^{-r} \, dx < \infty$. Then a direct calculation shows[11] that

$$\int_{M_1 \times N_1} \Xi(m_1 n)(1 + \sigma(m_1 n))^{-r} \Xi_{M_1}(m_1) \, d_Q(m_1) \, dm_1 \, dn = c.$$

From this and (2) we conclude that given $u, v \in \mathfrak{M}_1$, there exist $u_i \in \mathfrak{M}_1$ $(1 \leqq i \leqq s)$ such that, for all $f \in C_c^\infty(G)$,

$$\int_{M_1} |u(\Xi_{M_1} f_Q) v| \, dm_1 \leqq \sum_{1 \leqq i \leqq s} \int_{M_1} \Xi_{M_1} |u_i f v_i|_Q \, dm_1 \leqq c \sum_{1 \leqq i \leqq s} \mu_{u_i, v_i; r}(f).$$

Theorems 4.1.3 and 4.3.1 now give us what we want. The result for $l > 0$ follows from the following estimate [**6h**, §§42–43]: There exists $c > 0$ such that

(5) $$1 + \sigma(m_1 n) \geqq c(1 + \sigma(m_1)) \qquad (m_1 \in M_1, \, n \in N_1).$$

Lemma 1 yields at once

[11] In proving this we use the following generalization of (4.1.2): If τ is the function $km_1 n \mapsto \Xi_{M_1}(m_1) d_Q(m_1)^{-1}$ on G, then (cf. [**6h**, p. 101])

(*) $$\Xi(x) = \int_{K_1} \tau(x k_1) \, dk_1 \qquad (x \in G, \, K_1 = K \cap M_1).$$

One proves (*) by showing that the RHS is a spherical function satisfying (4.1.3).

LEMMA 2. *If $f\in\mathscr{C}(G)$, f_Q is well defined and lies in $\mathscr{C}(M_1)$; $f\mapsto f_Q$ is continuous from $\mathscr{C}(G)$ to $\mathscr{C}(M_1)$; and* (2) *is valid for all $f\in\mathscr{C}(G)$.*

LEMMA 3. *There exists a constant $c>0$ such that, for all $f\in C_c(G)$,*

$$(6)\qquad 'F_f(b)=c\,'F_{\bar f_Q}(b)\qquad (b\in L')$$

where $\bar f$ is defined by $\bar f(x)=\int_K f(kxk^{-1})\,dk$ $(x\in G)$.

Observe that L is a CSG of M_1. If $C=L_R$, these lemmas give us the required reduction, as L_I is a compact CSG of M. For Lemma 3 note that $(KN_1M)^-=\bar G$ and that $dk\,dn\,dm\sim d\bar x$. So after a simple calculation we find that there is a constant $c_1>0$ such that for all $f\in C_c(G)$, $b\in L'$,

$$\int_{\bar G} f(b^{\bar x})\,d\bar x=c_1\cdot\int_{N_1\times M}\bar f(m^*n^*)\,dn\,dm$$

where $m^*=mbm^{-1}$ and $n^*=m^*nm^{*-1}n^{-1}$. For fixed m, $n\mapsto n^*$ is an analytic diffeomorphism of N_1 onto itself, and $dn^*=|\det(\mathrm{Ad}(m^*)-1)_{\mathfrak{n}_1}|\cdot dn=d_Q(m^*)^{-1}\cdot|D_{\mathfrak{l}}(b)|^{1/2}\,dn$. This gives (6).

5.3. Proofs of the main theorems. The second and key step in the proof of Theorem 5.1.1 is the following lemma which we shall discuss in §5.4.

LEMMA 1. *Let L be compact and φ_L, the characteristic function of $(L')^G$. Then there exists $q\geq 0$ such that $\varphi_L\Xi(1+\sigma)^{-q}\in L^1(G)$. In particular, for any $a,b\in\mathfrak{G}$, $f\mapsto\|\varphi_L(afb)\|_1$ is a well-defined continuous seminorm on $\mathscr{C}(G)$.*

We shall indicate briefly how Theorem 5.1.1 (for compact L) follows from this. First we show that, for any $f\in C_c^\infty(G)$, $'F_f\in C^\infty(L')$ and satisfies (5.1.2) (cf. **[5d]**). Write $'\Delta={}'\Delta_I$, $\delta=\delta_I$, $\alpha(z)=e^\delta\circ\mu_{\mathfrak{g}/\mathfrak{l}}(z)\circ e^{-\delta}$ $(z\in\mathfrak{Z})$. We then obtain the following estimate: There exists $c>0$ such that, for all $f\in C_c^\infty(G)$,

$$(1)\qquad \int_{L'}|\alpha(z)'F_f|\,|'\Delta|\,db\leq c\,\|\varphi_L(zf)\|_1.$$

We now apply[12] Theorem 5.8.4 and Lemma 1 to conclude that $'F_f\in H^\infty(L')$, and that there is a continuous seminorm ν on $\mathscr{C}(G)$ for which $|'F_f(b)|\leq\nu(f)$, $\forall b\in L'$, $f\in C_c^\infty(G)$. Theorem 4.3.1 now gives the following estimate for some $q\geq 0$; this estimate leads easily to Theorem 5.1.1:

[12] If $f\in C_c^\infty(G)$, we may already conclude from Theorem 5.8.4 and Lemma 5.2.3 that for any CSG L_1, $'F_{f,L_1}$, vanishes outside a compact subset of L_1 and each of its derivatives is bounded on L_1'; Lemma 1 is needed only when $f\in\mathscr{C}(G)$.

(a) $$\sup_{b \in L'} |D(b)|^{1/2} \int_G \Xi(b^x)(1+\sigma(b^x))^{-q}\, dx < \infty.$$

It is interesting to observe that for general L (not necessarily compact or θ-stable) one can deduce the analogue of (2) from Theorem 5.1.1 with the help of Theorem 4.3.1. In fact, given L, there exists $c>0$ such that

(3) $$1+\sigma(b^x) \geqq c(1+\sigma(b)) \qquad (b \in L,\ x \in G)$$

as may be deduced from (5.2.5). Consequently we have the following estimate [**6h**, § 17]: there exists $q=q(L) \geqq 0$ such that for all $l \geqq 0$,

(4) $$\sup_{b \in L'} |D(b)|^{1/2}(1+\sigma(b))^l \int_{\bar{G}} \Xi(b^{\bar{x}})(1+\sigma(b^{\bar{x}}))^{-(q+l)}\, d\bar{x} < \infty.$$

We now consider Theorem 2. One may assume L to be compact and establish the results concerning $'F_f$ around each semiregular point of L; Theorem 2 would then follow from the fact that $'F_f$ and its derivatives are bounded in L'.[13] In other words, both Theorems 2 and 3 of § 5.1 would follow from a study of $'F_f$ in the neighborhood of an arbitrary semiregular point of L. Furthermore, in view of the continuity of the map $f \mapsto {'F_f}$, it is enough to do this for f lying in $C_c^\infty(G)$ or a dense subspace thereof. This observation enables us to come down to the case when $\mathfrak{g}$ is semisimple.

Fix a semiregular $b \in L$. We select a system $(\mathfrak{u}, V)$ adapted to b (we shall use the notation of §§ 2, 3). Now there are invariant functions $g \in C^\infty(G)$ such that $g=1$ in a neighborhood of b in G and supp $g \subseteq V$. Consequently it is enough to study $'F_f$ in a neighborhood of b in L for $f \in C_c^\infty(V)$.

Let $\tilde{G}=G/M_b$, $M_b^*=M_b/M_b \cap L$ and let $\tilde{x}, d\tilde{x}, y^*, dy^*$ have their obvious meanings. Given $f \in C_c^\infty(V)$, $x \in G$, and $H \in {'\mathfrak{l}} \cap \mathfrak{u}$, we define $\tilde{f}(x:H)=f((b \exp H)^x)$. Then there exists a constant $c>0$ such that, for all H, f as above,

(5) $$\int_{\bar{G}} f((b \exp H)^{\bar{x}})\, d\bar{x} = c \int_{\tilde{G}} \left(\int_{M^*_b} \tilde{f}(x:H^{y^*})\, dy^* \right) d\tilde{x};$$

[13] The principle we are appealing to can be formulated as follows. Let X be a real Hilbert space of finite dimension d; B, an open ball with center at 0; $W_1, \ldots, W_n$ distinct linear subspaces of dimension $d-1$; and f, a C^∞ function on $B'=B \backslash \bigcup_i W_i$, such that each derivative of f is bounded on B'. Suppose $0 \leqq k \leqq \infty$, and that for each i $(1 \leqq i \leqq n)$ and each $x \in B \cap (W_i \backslash \bigcup_{j \neq i} W_j)$, there is a neighborhood N_x of x and a function ψ_x of class C^k on N_x such that $\psi_x=f$ on $N_x \cap B'$. Then there is a function ψ of class C^k on B such that $\psi=f$ on B'.

here the inner integral on the RHS depends only on $\tilde{x}$ and defines a function belonging to $C_c^\infty(\tilde{G})$, and it is this function that is integrated over $\tilde{G}$. Since $\dim[\mathfrak{m}_b, \mathfrak{m}_b]=3$, the orbital integrals over M_b^* in (5) are accessible through explicit calculation. Theorems 2 and 3 follow without much difficulty from these calculations [**6c**, §§7–10].

5.4. Continuity over $\mathscr{C}(G)$ of L^1-norms on the elliptic set. Proof of Lemma 5.3.1. Lemma 5.3.1 would follow if we establish the following: there exists $c>0$ such that, for almost all $x\in G$,

$$\int_K \varphi_L(xk)\,dk \leqq c\Xi(x) \qquad (L \text{ compact}). \tag{1}$$

We shall obtain (1) as a consequence of the following more general results [**6g**, Theorems 4 and 5]:

Let L be any θ-stable CSG; then the function $\varphi_L: x\mapsto |D_{\mathfrak{l}}(x)|^{-1/2}$ is locally summable on G, and there is a constant $c>0$ such that, for almost all $x\in G$,

$$\int_K \varphi_L(xk)\,dk \leqq c\Xi(x). \tag{2}$$

The function $|D|^{-1/2}$ is locally summable[14] on G [**6e**, §28]; φ_L is locally summable, as $\sup_{x\in G'}(|D(x)|^{1/2}\,\varphi_L(x))<\infty$. The LHS of (2) is thus finite for almost all $x\in G$. We prove (2) by induction on $\dim(G)$. Let I_c^+ be the set of all $f\in C_c(G)$ that are spherical and $\geqq 0$. For any invariant locally summable function Θ on G write $\Theta_0(x)=\int_K \Theta(xk)\,dk$; Θ_0 is locally summable, spherical, and $\int_G \Theta f\,dx=\int_G \Theta_0 f\,dx$, $\forall f\in I_c^+$; if Θ is $\mathfrak{Z}$-finite, $\Theta_0\in C^\infty(G)$. We may also assume $G={}^\circ G$.

Let $\dim(L_R)>0$ and $\varphi_{L_I}^M$ be the characteristic function of $(L_I')^M$, M being as in §5.2 with $C=L_R$. Then, by Lemma 5.2.3, there exists $c_1>0$ such that, for all $f\in I_c^+$,

$$\int_G \varphi_L f\,dx \leqq c_1 \int_{M\times L_R} \varphi_{L_I}^M(m)\, f_Q(mb)\,dm\,db. \tag{3}$$

Estimating $\varphi_{L_I}^M$ by the induction hypothesis and using (5.2.3), we find that for some constant $c_2>0$ and all $f\in I_c^+$, the RHS of (3) is $\leqq c_2\int_G \Xi f\,dx$. We are thus left with the case of compact L. Since both sides of (2) depend only on $\mathrm{Ad}(x)$ we may assume that $G\subseteq G_c$ where G_c is complex, semisimple, and simply connected. As L is now connected, $(L')^G$ is contained in the component of 1 of G, and so we may

[14] This follows from the fact that $\int_{L_1}|'F_{f,L_1}|\,db<\infty$ for all $f\in C_c^\infty(G)$ and all CSG's L_1 (cf. footnote 12).

suppose that G is connected. We write $L=B$ and use notation of §3.

The proof of (1) is difficult because for a given $x\in G$ it is not a simple matter to determine the set of $k\in K$ for which xk is elliptic. If Θ_π is the character of a finite-dimensional representation π of G, $|\Theta_\pi(y)|\leqq \dim(\pi)$ for all elliptic y; and by considering all possible π it is not difficult to establish the following weaker result as a first step towards (1) [**6g**, Lemma 42];

$$\lim_{x\to\infty} (\varphi_B)_0(x)=0. \tag{4}$$

The main idea in Harish-Chandra's proof of (1) is to use the distribution $\Theta^*=\Theta^*_\delta$ in place of Θ_π; here $\delta=\frac{1}{2}\sum_{\alpha\in P}\alpha$ and Θ^*_δ is defined by (3.6.1). $\Theta^*=1$ on $(B')^G$; *and, unlike* Θ_π, *it is of slow growth on all the CSG's of* G.

A simple calculation shows that if $\{X_1,\dots,X_r\}$ and $\{X_{r+1},\dots,X_n\}$ are orthonormal bases of $\mathfrak{k}$ and $\mathfrak{p}$ respectively, and if $q=X_1^2+\cdots+X_n^2$, then $q\Theta^*=0$, whence $q\Theta^*_0=0$ also. Further, the estimates of Theorem 3.6.1 imply that, for some constant $c>0$ and almost all x,

$$|\Theta^*_0(x)-(\varphi_B)_0(x)|\leqq c\sum_{1\leqq i\leqq s}(\varphi_{L_i})_0(x), \tag{5}$$

$L_1,\dots,L_s$ being a complete system of noncompact CSG's of G. From (4), (5) and the induction hypothesis we find that $\lim_{x\to\infty}\Theta^*_0(x)=0$. The maximum principle for the (second degree) elliptic operator q implies that $\Theta^*_0=0$. But then (1) follows from (5) and the induction hypothesis.

It remains to sketch a proof of (4). Let Σ be the simple system of roots of $(\mathfrak{g},\mathfrak{a})$. Extend $\mathfrak{a}$ to a CSA $\mathfrak{h}$ and let S be the simple system corresponding to an ordering of the roots of $(\mathfrak{g},\mathfrak{h})$ that is compatible with $\mathfrak{a}^+$. If (4) were false, we can find $\gamma>0$, $F\subsetneqq\Sigma$, and $\{H_n\}_{n\geqq 1}$ from $\mathrm{Cl}(\mathfrak{a}^+)$ such that (i) $\lambda(H_n)=O(1)$ or $\to+\infty$ according as $\lambda\in F$ or $\in\Sigma\backslash F$, and (ii) $(\varphi_B)_0(a_n)\geqq\gamma>0$ for all n, where $a_n=\exp H_n$. Let S' be the set of all $\beta\in S$ whose restrictions to $\mathfrak{a}$ are in $\boldsymbol{R}\cdot F$, and let π be the irreducible representation of G whose highest weight Λ is such that $\langle\Lambda,\beta\rangle$ is $=0$ or >0 according as $\beta\in S'$ or $\in S\backslash S'$. If Γ is the set of $\mu\in\mathfrak{h}^*$ of the form $\sum_{\beta\in S}c(\beta)\,\beta$ with $c(\beta)\geqq 0$ for all β and $c(\beta)>0$ for some $\beta\in S\backslash S'$, then it can be shown [**6g**, §22] that $\Lambda\in\Gamma$, and that, for any weight $\Lambda'\neq\Lambda$ of π, $\Lambda-\Lambda'\in\Gamma$. Let $u_1,\dots,u_p$ be a basis of weight vectors of π, with Λ_i as the weight of u_i and $\Lambda_1=\Lambda$; and let $(a_{ij}(x))$ be the matrix of $\pi(x)$ $(x\in G)$ in this basis. If $K_n=\{k:k\in K,\ a_nk \text{ is elliptic}\}$, the inequality $|\Theta_\pi(a_nk)|\leqq p$ $(k\in K_n)$ then implies that

$$|a_{11}(k)|\leqq p\exp\{-\Lambda(H_n)\}+\sum_{2\leqq i\leqq p}|a_{ii}(k)|\exp\{-(\Lambda-\Lambda_i)(H_n)\}\qquad(k\in K_n).$$

So $K_n\subseteq\{k:|a_{11}(k)|\leqq\tau_n\}$ where $\tau_n\to 0$. Since $\{k:k\in K,\ a_{11}(k)=0\}$ has measure zero, we find that $(\varphi_B)_0(a_n)=\int_{K_n}dk\to 0$, a contradiction.

It is clear from our discussion so far that the map $f \mapsto \bar{f}_Q$ establishes an intimate connexion between problems of invariant analysis on G and those on M_1. This connexion actually goes much deeper than the above results would seem to suggest, and it may be of some interest to look into this a little more closely. We have $G = K \exp(\mathfrak{m} \cap \mathfrak{p}) CN_1$, and for $x \in G$ let $\mu(x)$, $c(x)$ and $n(x)$ be the components of x in $\exp(\mathfrak{m} \cap \mathfrak{p})$, C and N_1 respectively. Then we have the following formula, valid for all continuous functions f on G with $\|\Xi^{-1}(1+\sigma)^r f\|_\infty < \infty$ for all $r \geqq 0$:

$$(6) \qquad (\bar{f^x})_Q = \int_K [(\bar{f}^{k^{-1}})_Q]^{\mu(xk)} \, dk \qquad (x \in G).$$

Let us now write T_Q for the map which is dual to the map $f \mapsto \bar{f}_Q$. Then, (6) shows that for any tempered invariant distribution τ on M_1, $T_Q(\tau)$ is a tempered invariant distribution on G. Moreover, for such τ,

$$(7) \qquad zT_Q(\tau) = T_Q(\mu_{\mathfrak{g}/\mathfrak{m}_1}(z)\, \tau) \qquad (z \in \mathfrak{Z}).$$

In particular, if τ is an eigendistribution on M_1, $T_Q(\tau)$ has the same property on G. Explicit calculations show that if τ is the character of an irreducible unitary representation π_Q of Q that is trivial on N_1, $T_Q(\tau)$ is the character of the representation of G that is induced by π_Q.

5.5. Tempered invariant eigendistributions. In this and the next two paragraphs we shall use the foregoing theory to study various questions of harmonic analysis on G. For simplicity we assume in this paragraph that G is as in §§2 and 3 and use the notation therein.

THEOREM 1. *Let Θ be an invariant $\mathfrak{Z}$-finite distribution on G. Then the following statements are equivalent:*

(i) *Given any CSG L, there exist $C = C_L > 0$, $q = q_L \geqq 0$ such that $|\Phi_{L,P}(a)| \leqq C(1+\sigma(a))^q$ for all $a \in L'$.*

(ii) *There exist $C > 0$, $q \geqq 0$ such that $|\Theta(x)| \leqq C|D(x)|^{-1/2}(1+\sigma(x))^q$ for all $x \in G'$.*

(iii) *Θ is tempered.*

If these conditions are satisfied, then

$$(1) \qquad \Theta(f) = \int_G \Theta(x) f(x)\, dx \qquad (f \in \mathscr{C}(G))$$

the integral converging absolutely.

From (5.5.3) we see that (i)⇒(ii). Further, (5.3.4) shows that, for some $q \geqq 0$,

(2) $$\int_G |D(x)|^{-1/2}\Xi(x)(1+\sigma(x))^{-q}\,dx<\infty.$$

The implication (ii)⇒(iii) as well as the last assertion follows at once from (2). For the proof that (iii)⇒(i) see [**6h**, § 19]; the main argument here is similar to but more delicate than Lemma 3.5.1.

COROLLARY 2. *The distributions* Θ_λ ($\lambda\in\mathcal{L}'$) *are tempered.*

COROLLARY 3. *Let* Θ *be an invariant eigendistribution such that the corresponding eigenhomomorphism is regular. Then* Θ *is tempered if and only if*

(3) $$\sup_{x\in G'}|D(x)|^{1/2}\,|\Theta(x)|<\infty.$$

5.6. The relation $(\varpi F_f)(1)=cf(1)$. If G is compact and B is a maximal torus, then $F_{f,B}$ is a C^∞ function on B and its harmonic analysis leads at once to the Plancherel formula. It is tempting to suppose that the same method would yield significant results even in the noncompact case. While this turns out to be substantially the case, complications arise because the functions F_f are no longer C^∞, and the jumps of these functions and their derivates cannot be ignored. Following Harish-Chandra we shall formulate Theorem 1 below as an important step towards the Plancherel formula ([**5d**], [**6h**]; see also [3]). In §7 we shall see that it leads to a complete determination of the discrete series.

A CSG L will be called *fundamental* if L_I has the maximum possible dimension. It is known that any two fundamental CSG's are conjugate in G and that L is fundamental if and only if $(\mathfrak{g},\mathfrak{l})$ has no real roots [**5b**, §8].

THEOREM 1. *Let G be of class $\mathscr{H}$. Fix a CSG L and let $\varpi_L=\prod_{\alpha\in P_L}H_\alpha$. Then, if L is not fundamental, $('\varpi_L' F_{f,L})(1)=0$ for all $f\in\mathscr{C}(G)$.*

Suppose L is fundamental. Let $q=\frac{1}{2}(\dim G/K-\mathrm{rk}\,G+\mathrm{rk}\,K)$.[15] *Then there is a constant $c>0$ such that for all $f\in\mathscr{C}(G)$,*

(1) $$f(1)=(-1)^q\,c('\varpi_L' F_{f,L})(1).$$

If G is as in §3, $(\varpi F_{f,L})(1)=0$ *for all $f\in\mathscr{C}(G)$ for nonfundamental L, while for fundamental L,*

(2) $$f(1)=(-1)^q\,c(\varpi_L F_{f,L})(1)\qquad(f\in\mathscr{C}(G)).$$

We discuss this when G is as in §3. If L is not fundamental and α is any real root of $(\mathfrak{g},\mathfrak{l})$, we observe that $F_{f,L}$ is invariant under s_α on $L'\cap\exp\mathfrak{l}$. So $(\varpi_L F_{f,L})^{s_\alpha}=-(\varpi_L F_{f,L})$ on $\exp\mathfrak{l}$, implying $(\varpi_L F_{f,L})(1)=0$. Let L be fundamental and let

[15] q is an integer $\geqq 0$.

(3) $$\Gamma(f)=(\varpi_L F_{f,L})(1) \qquad (f\in\mathscr{C}(G)).$$

Γ is an invariant tempered distribution. A simple argument shows that the support of Γ is contained in the set of unipotent elements of G. We may thus transfer Γ to the Lie algebra. The theorem then follows from the work of §5.9.

5.7. Cusp forms. Let G be a group of class $\mathscr{H}$. Following Harish-Chandra [**6i**, p. 538] we shall define a *cusp form* on G to be any $f\in\mathscr{C}(G)$ such that, for any psgrp $Q=MCN_1\neq G$,

(1) $$\int_{N_1} f(xn)\,dn=0 \qquad (x\in G).$$

${}^\circ\mathscr{C}(G)$ denotes the set of all cusp forms on G. It is a closed subspace of $\mathscr{C}(G)$ invariant under all translations; the estimates (4.2.3) and (5.2.4) imply easily that it is a two-sided ideal in $\mathscr{C}(G)$. The importance of ${}^\circ\mathscr{C}(G)$ for harmonic analysis on G lies in the fact that its closure in $L^2(G)$ is precisely the closed linear span of all the subspaces of $L^2(G)$ that are irreducibly invariant under the regular representation. We shall prove this in §7 (heuristically, (1) expresses the condition that f be orthogonal to the principal series of representations associated with Q). In this paragraph we shall formulate some of the properties of cusp forms. Theorem 1 below is especially noteworthy; it reveals the real reason why the harmonic analysis of a cusp form involves only the discrete series.

THEOREM 1. *Let $f\in{}^\circ\mathscr{C}(G)$. Then $'F_{f,L}=0$ for any noncompact CSG. If L is compact, $'F_{f,L}$ extends to a C^∞ function on L.*

The relation (5.2.6) (extended to $\mathscr{C}(G)$) gives the first assertion. If L is compact, Theorem 5.1.3 implies that $'F_{f,L}$ is C^∞ around each regular or semiregular point; this implies the second assertion (cf. footnote 13).

THEOREM 2. *Let f be a $\mathfrak{Z}$-finite function in $\mathscr{C}(G)$. Then $f\in{}^\circ\mathscr{C}(G)$. Moreover, ${}^\circ\mathscr{C}(G)$ is the closure of the space of $\mathfrak{Z}$-finite K-finite functions in $\mathscr{C}(G)$.*

For the second part see §7. For the first let $f\in\mathscr{C}(G)$ be $\mathfrak{Z}$-finite and $Q=MCN_1\neq G$ a psgrp. Then for all $m\in M$, $a\mapsto f_Q(ma)$ is $\mathfrak{C}$-finite on C, hence $\equiv 0$. So $f_Q\equiv 0$. This implies easily that $f\in{}^\circ\mathscr{C}(G)$.

THEOREM 3. *${}^\circ\mathscr{C}(G)\neq\{0\}\Leftrightarrow G$ has a compact CSG.*

Suppose there exists $f\in{}^\circ\mathscr{C}(G)$, $f\neq 0$. If no CSG is compact, $'F_{f,L}=0$ for all L. Hence $f(1)=0$ by the work of §5.6. Replacing f by its translates, $f(x)=0$ for all x. The converse follows from Theorems 6.1.2 and 6.1.3.

5.8. Appendix. Some estimates in a classical setting. The results described in this paragraph assert that in certain situations one can estimate sup norms by L^p-norms ($1 \leqq p < \infty$). These estimates are closely related to those obtained by Harish-Chandra in the context of Theorem 3 of [**6e**].

We fix a real Hilbert space V of finite dimension d; S is the symmetric algebra over V_c; $S_0 \subseteq S$ a subalgebra such that $1 \in S_0$ and S is a finite module over S_0. If $U \subseteq V$ is open and $w \in C^\infty(U)$ is >0, $H^p(U, w, S_0)$ denotes the space of all $f \in C^\infty(U)$ for which $\|\eta f\|_{p,w} = (\int_U |\eta f|^p w\, dx)^{1/p} < \infty$, $\forall \eta \in S_0$ (we write $H^p(U, w)$ when $S_0 = S$); $H^\infty(U)$ is the space of all $f \in C^\infty(U)$ for which $\|\eta f\|_\infty < \infty$, $\forall \eta \in S$. We topologize these spaces by the corresponding seminorms. $B(x, a)$ is the closed ball of center x and radius $a > 0$.

LEMMA 1. *Fix U, w as above and a real function ε on U such that $0 < \varepsilon(x) \leqq 1$ and $B(x, \varepsilon(x)) \subseteq U$ for all $x \in U$. Put $\omega(x) = \inf_{y \in B(x, \varepsilon(x))} w(y)$. Then there exists an integer $b \geqq 0$, and, for each $\xi \in S$, a continuous seminorm ν_ξ on $H^p(U, w, S_0)$ such that*

$$(1) \qquad |f(x;\xi)| \leqq \varepsilon(x)^{-b}\, \omega(x)^{-1/p}\, \nu_\xi(f) \qquad (x \in U, f \in H^p(U, w, S_0)).$$

In particular $H^p(U, w, S_0)$ is Fréchet.

As S is a finite module over S_0 we may prove (1) with some $b = b(\xi)$. Let Δ be the Laplacian of V; $D = 1 - \Delta$; and k_r ($2r > d$), the tempered fundamental solution of D^r. $k_r \in C^{(2r-d-1)}(V)$ is C^∞ on $V \backslash \{0\}$, and, for $\eta \in S$ of degree n, $(\eta k_r)(x) = O(\|x\|^{-2n})$ for $x \to 0$. Fix $\xi \in S$, let $s = d + 1 + \deg(\xi)$, and choose $\eta_1, \ldots, \eta_m \in S_0$ such that $(D^s)^m = \sum_{1 \leqq j \leqq m} \eta_j (D^s)^{m-j}$. Then, for all $g \in C_c^\infty(V)$ and all $x \in V$,

$$(*) \qquad g(x;\xi) = \sum_{1 \leqq j \leqq m} \int_V k_{js}(x-y;\xi)\, g(y;\eta_j)\, dy.$$

We now select "localizing" functions $\psi_x \in C_c^\infty(U)$ for $x \in U$ such that (i) $0 \leqq \psi_x \leqq 1$ and $\psi_x(y) = 0$ or 1 according as $\|y - x\| > \frac{3}{4}\varepsilon(x)$ or $< \frac{1}{4}\varepsilon(x)$, and (ii) if $\zeta \in S$, there exists $c_\zeta > 0$ with $|(\zeta\psi_x)(y)| \leqq c_\zeta \varepsilon(x)^{-\deg(\zeta)}$ for all $y \in V$, $x \in U$ (cf. [**14b**, §3]). On the other hand there exist $\zeta_q, \sigma_{jq} \in S$ with the σ_{jq} having zero constant terms such that for all $h \in C^\infty(V)$, $\eta_j \circ h = h\eta_j + \sum_{1 \leqq q \leqq M} (\sigma_{jq} h)\, \zeta_q$ ($1 \leqq j \leqq m$). Taking $g = f\psi_x$ in $(*)$ we obtain, for all $f \in C^\infty(U)$, $x \in U$,

$$f(x;\xi) = \sum_{1 \leqq j \leqq m} \int_{B(x,\varepsilon(x))} k_{js}(x-y;\xi)\, \psi_x(y)\, f(y;\eta_j)\, dy$$

$$+ \sum_{1 \leqq j \leqq m} \sum_{1 \leqq q \leqq M} \int_{\frac{1}{4}\varepsilon(x) \leqq \|y-x\| \leqq \frac{3}{4}\varepsilon(x)} F_{j,q,x}(y;\zeta_q^\dagger)\, f(y)\, dy;$$

here $F_{j,q,x}(y)=k_{js}(x-y;\xi)\,\psi_x(y;\sigma_{jq})$ and $\zeta_q^\dagger$ is the adjoint of ζ_q. The estimates for k_r and $\zeta\psi_x$ now lead to (1).

THEOREM 2. *Let $\lambda_j\in V_c^*$ be nonzero $(1\leqq j\leqq q)$; $V'=\{x:x\in V,\ \lambda_j(x)\neq 0,\ \forall j\}$; U, a union of components of V'; and suppose that, for some $c>0$, $r\geqq 0$,*

$$(2)\qquad w(x)\geqq c\left(1+\max_j|\lambda_j(x)|^{-1}\right)^{-r}\qquad (x\in U).$$

Then $H^p(U,w,S_0)\subseteq H^\infty(U)$, the inclusion map being continuous.

It is easy to come down to the case where the λ_j are real on V and U is the set where they are all >0; this can then be handled with the help of Lemma 1 (cf. [**14b**, §3]). We mention two situations where the conditions of Theorem 2 are satisfied:

(i) $w(x)=\prod_j|\lambda_j(x)|^{a_j}$ ($a_j>0$ are constants), and

(ii) the λ_j are real on V, U is the set where they are all >0, and $w=\prod_j(1-e^{-\lambda_j})$.

Let $\mathscr{C}(U)$ be the Schwartz space of U.

COROLLARY 3. *Let $w(x)=\prod_j|\lambda_j(x)|^{a_j}$ ($a_j>0$ constant). Fix a subalgebra P_0 of the algebra P of all polynomials on V_c such that $1\in P_0$ and P is a finite module over P_0. Then $\mathscr{C}(U)=\{f:gf\in H^p(U,w,S_0),\ \forall g\in P_0\}$, and the topology of $\mathscr{C}(U)$ coincides with that induced by the seminorms $f\mapsto\|\xi(gf)\|_{p,w}$ ($\xi\in S_0$, $g\in P_0$).*

Let T be a compact Lie group whose identity component is abelian and has V as Lie algebra. Let $\Omega=\prod_{1\leqq j\leqq q}|\chi_j-1|$ where the χ_j are one-dimensional characters of T that are $\not\equiv 1$ on any component of T. Put $T'=\{b:b\in T,\ \Omega(b)\neq 0\}$ and define $H^p(T',\Omega,S_0)$ and $H^\infty(T')$ in the obvious way. These are Fréchet spaces.

THEOREM 4. *$H^p(T',\Omega,S_0)=H^\infty(T')$ as Fréchet spaces.*

We fix $f\in H^p(T',\Omega,S_0)$, $b\in T$, $\xi\in S$, and show that for some neighborhood N of b, $\|\xi f\|_{\infty,N\cap T'}<\infty$. We may assume that $\chi_j(b)=1$ for all j. Let $\lambda_j\in V^*$ be such that $\chi_j(b\exp x)=\exp\{(-1)^{1/2}\lambda_j(x)\}$ $(x\in V)$; write $w(x)=\prod_j|\lambda_j(x)|$ and V' as before. Clearly there exist $\alpha,\beta>0$ sufficiently small such that $|\Omega(b\exp x)|\geqq\beta w(x)$ if $\|x\|\leqq\alpha$, and $\varphi(x\mapsto f(b\exp x))$ lies in $H^p(B(0,2\alpha)\cap V',w,S_0)$. We now fix a component V^+ of V' and apply Lemma 1 with U as the (convex) open set $V^+\cap B(0,\alpha)$ and $\varepsilon(x)=\frac{1}{2}\min(2\alpha-\|x\|,\ \min\|\lambda_j\|^{-1}|\lambda_j(x)|)$. As $\omega(x)\geqq(\varepsilon(x)/2)^q$, there exists an integer $l\geqq 0$ such that, for all $\eta\in S$, $\sup_{x\in U}\varepsilon(x)^l|(\eta\varphi)(x)|<\infty$. This implies that $\|\xi\varphi\|_{\infty,U}<\infty$. In fact, fix $x_0\in U$ and write $\varphi_x(t)=\varphi(x_t;\xi)$ where $0\leqq t\leqq 1$, $x_t=(1-t)x+tx_0$ $(x\in U)$. Then there exists $\gamma>0$ such that $\varepsilon(x_t)\geqq\gamma t$ for all $x\in U$, $0\leqq t\leqq 1$, and so we can find constants $L_m>0$ $(m=0,1,\dots)$ such that $|\varphi_x^{(m)}(t)|\leqq L_m t^{-l}$, $\forall m\geqq 0$, $x\in U$, $0<t\leqq 1$. The arguments of [**14b**, §3] now lead to the desired conclusion.

5.9. Appendix. Tempered invariant eigendistributions on a semisimple Lie algebra. In this appendix we shall describe some important results in the theory of tempered invariant eigendistributions on a semisimple Lie algebra $\mathfrak{g}$. The main references are [**5b**], [**5c**], [**6c**].

Let $\mathfrak{g}$ be as above, and G, the adjoint group of $\mathfrak{g}$. Fix a CSA $\mathfrak{l}$ with CSG L. Let $G^* = G/L$; $x^* = xL$ $(x \in G)$; and dx^*, the invariant measure on G^*. Let P be a positive system of roots of $(\mathfrak{g}, \mathfrak{l})$; $\pi = \prod_{\alpha \in P} \alpha$; $\varpi = \prod_{\alpha \in P} H_\alpha$. For $p \in I(\mathfrak{g})$, $p_\mathfrak{l}$ is its restriction to $\mathfrak{l}$. We often write $\partial(q)$ for the differential operator corresponding to $q \in S(\mathfrak{g}_c)$. Write $\varepsilon_R(H) = \operatorname{sgn} \prod_{\alpha \in P, \alpha \text{ real}} \alpha(H)$ $(H \in \mathfrak{l}')$. $\mathscr{C}(\mathfrak{g})$ is the Schwartz space of $\mathfrak{g}$. We define the Fourier transform map on $\mathscr{C}(\mathfrak{g})$ and its dual in the usual way (using $\langle \cdot, \cdot \rangle$).

THEOREM 1. *For any $f \in \mathscr{C}(\mathfrak{g})$ and $H \in \mathfrak{l}'$,*

$$\psi_f(H) = \varepsilon_R(H)\, \pi(H) \int_{G^*} f(H^{x^*})\, dx^* \tag{1}$$

is well defined, the integral converging absolutely. $\psi_f \in \mathscr{C}(\mathfrak{l}')$ and $f \mapsto \psi_f$ is a continuous map of $\mathscr{C}(\mathfrak{g})$ into $\mathscr{C}(\mathfrak{l}')$. Moreover

$$\psi_{\partial(p) f} = \partial(p_\mathfrak{l})\, \psi_f \qquad (p \in I(\mathfrak{g}),\ f \in \mathscr{C}(\mathfrak{g})). \tag{2}$$

THEOREM 2. *Let $\mathfrak{l}'(I)$ be the set of all $H \in \mathfrak{l}$ where no singular imaginary root is 0. Fix $f \in \mathscr{C}(\mathfrak{g})$. Then ψ_f extends to a C^∞ function on $\mathfrak{l}'(I)$. Suppose $H \in \mathfrak{l}$ and $S_I(H)$ is the set of all singular imaginary roots in P vanishing at H. If $\zeta \in S(\mathfrak{l}_c)$ is such that $\zeta^{s_\beta} = -\zeta$ for all $\beta \in S_I(H)$, then $\partial(\zeta)\, \psi_f$ extends as a continuous function around H. In particular, $\partial(\varpi)\, \psi_f$ extends to a continuous function on $\mathfrak{l}$.*

The proofs are similar to those of the corresponding results on the group. In Theorem 1, the estimates furnished by Theorem 2 and Corollary 3 of §5.8 enable one to handle the convergence problems. It follows from these results that for any $H \in \mathfrak{l}'$ the invariant measure on H^G is tempered. We write

$$\sigma_H(f) = \int_{G^*} f(H^{x^*})\, dx^* \qquad (f \in \mathscr{C}(\mathfrak{g})). \tag{3}$$

THEOREM 3. *The distribution $\hat{\sigma}_H$ is invariant and $\partial(p)\, \hat{\sigma}_H = p((-1)^{1/2} H)\, \hat{\sigma}_H$ $(p \in I(\mathfrak{g}),\ H \in \mathfrak{l}')$. Suppose $\mathfrak{l}_1 \subseteq \mathfrak{g}$ is a CSA, and y is an element of the adjoint group of $\mathfrak{g}_c$ such that $\mathfrak{l}_c^y = (\mathfrak{l}_1)_c$. Then there are uniquely defined locally constant functions $c_s(\cdot : \cdot)$ $(s \in W_{L_c})$ on $\mathfrak{l}' \times \mathfrak{l}_1'$ such that (writing $\pi_1 = \pi \circ y^{-1}$)*

$$\pi_1(H_1)\, \pi(H)\, \hat{\sigma}_H(H_1) = \sum_{s \in W_{L_c}} \varepsilon(s)\, c_s(H : H_1)\, e^{i\langle H_1, (sH)^y \rangle} \qquad (H \in \mathfrak{l}',\ H_1 \in \mathfrak{l}_1'). \tag{4}$$

Since $\hat{\sigma}_H$ is an invariant eigendistribution one obtains (4) with $c_s(H:\cdot)$ locally constant on $\mathfrak{l}_1'$ for all $H \in \mathfrak{l}'$. Also it is easy to show that $c_s(\cdot:H_1)$ is C^∞ for all $H_1 \in \mathfrak{l}_1'$. The relations (2) are then used to conclude that $c_s(\cdot:H_1)$ is locally constant on $\mathfrak{l}'$ for all $H_1 \in \mathfrak{l}_1'$.

THEOREM 4. *If* $\mathfrak{l}$ *is not fundamental,* $(\partial(\varpi)\psi_f)(0)=0$ *for all* $f \in \mathscr{C}(\mathfrak{g})$. *For fundamental* $\mathfrak{l}$, *there exists a constant* $c>0$ *such that, with* q *as in Theorem* 5.6.1,

$$cf(0)=(-1)^q(\partial(\varpi)\psi_f)(0) \qquad (f \in \mathscr{C}(\mathfrak{g})). \tag{5}$$

The first assertion is proved as in Theorem 5.6.1. Let $\mathfrak{l}$ be fundamental and $\gamma(f)=(\partial(\varpi)\psi_f)(0)$ $(f \in \mathscr{C}(\mathfrak{g}))$. It follows from (2) that $\hat{\gamma}$ is $I(\mathfrak{g})$-finite, and from (4) that $\hat{\gamma}$ is locally constant on $\mathfrak{g}'$ must be a constant [**6d**, §10]. So, for some constant c_1, $\gamma(f)=c_1 f(0)$, $\forall f \in \mathscr{C}(\mathfrak{g})$. The proof that $(-1)^q c_1$ is real and >0 is, however, delicate. It depends on the construction (based on some work of de Rham) and properties of invariant fundamental solutions to the differential operators $\partial(\omega)^m$ $(m \geq 1)$ where ω is the Casimir element in $I(\mathfrak{g})$ [**6c**, §§11–13].

6. Behaviour at infinity of eigenfunctions

6.1. Outline of the main results. We shall now take up the problem of showing that the Fourier components with respect to K of the distributions Θ_λ lie in $L^2(G)$. As these are tempered (Corollary 5.5.2), we may subsume this under the general problem of determining the behaviour, at infinity on G, of tempered, K-finite, $\mathfrak{Z}$-finite functions.

Let G be a group of class $\mathscr{H}$. For any finite-dimensional double representation $\tau=(\tau_1, \tau_2)$ of K in U, let $\mathscr{A}(G:\tau)$ be the space of all tempered $\mathfrak{Z}$-finite functions $f \in C^\infty(G:\tau)$. As $G=K\,\mathrm{Cl}(A^+)\,K$ our problem is that of determining, for $f \in \mathscr{A}(G:\tau)$, the behaviour of $f(a)$ as $a \to \infty$ in $\mathrm{Cl}(A^+)$. Since $\lambda(\log a)$ may not tend to ∞ for all $\lambda \in \Sigma$ (the set of simple roots of $(\mathfrak{g}, \mathfrak{a})$), we put for each $F \subsetneq \Sigma$, $\beta_F(H) = \min_{\lambda \in \Sigma \setminus F} \lambda(H)$ $(H \in \mathfrak{a})$, and study for arbitrary F how $f(a)$ behaves when $\beta_F(\log a) \to \infty$ $(\mathrm{Cl}(A^+) \ni a \underset{F}{\to} \infty$ in symbols).

Fix F and let $P_F=M_{1F}=M_F A_F N_F$ be the corresponding standard psgrp; let $M_{1F}^+ = \bigcup_{\mathfrak{a} \ni H:\, \beta_F(H)>0} K_F(\exp H)\,K_F$. Then for any $z \in \mathfrak{Z}$, there is a differential operator E_z on M_{1F}^+ such that (i) $zg=(d_F^{-1} \circ \mu_F(z) \circ d_F)\, g + E_z g$ on M_{1F}^+ for all $g \in C^\infty(G:\tau)$, and (ii) as functions of (m, a) $(m \in M_F, a \in A_F)$, the coefficients of E_z go to zero when $a \underset{F}{\to} \infty$. Thus, if $f \in C^\infty(G:\tau)$ is any tempered eigenfunction $(zf=\chi(z)f$, $\forall z \in \mathfrak{Z})$, the function $m_1 \mapsto d_F(m_1) f(m_1)$ on M_{1F}^+ satisfies certain differential equations which are *perturbations* of the equations $\mu_F(z)\,h=\chi(z)\,h$ $(z \in \mathfrak{Z})$. It follows from this that for a suitable tempered solution f_F of the unperturbed equations one can approximate $d_F(ma)\,f(ma)$ by $f_F(ma)$ when $A_F \ni a \underset{F}{\to} \infty$. The knowledge of these f_F, together with estimates for $|d_F(ma)\,f(ma)-f_F(ma)|$, then yield a complete picture of the asymptotic behaviour of f. When generalized so as to take care of $\mathfrak{Z}$-finite functions, this method leads to the following theorems which are the main results of the theory. In essence this is Harish-Chandra's method (cf. [**6h**, §§27–31], [**5f**, §§2–8]; cf. also [**14**]); f_F is the so-called *constant term of f along the psgrp P_F*.

THEOREM 1. *Let G be as above and fix $f \in \mathscr{A}(G:\tau)$. Then, for each $F \subsetneq \Sigma$, there is a unique $f_F \in \mathscr{A}(M_{1F}, \tau_F)$ $(\tau_F = \tau \mid K_F)$ such that, for all $m \in M_{1F}$,*

$$(1) \qquad |d_F(m \exp tH) f(m \exp tH) - f_F(m \exp tH)| \to 0 \qquad (t \to +\infty, H \in \mathfrak{a}_F^+);$$

$\mu_F(z) f_F = 0$ for all $z \in \mathfrak{Z}$ for which $zf = 0$. For any $\kappa > 0$ let

$$(2) \qquad A^+(F:\kappa) = \{a : a \in \mathrm{Cl}(A^+), \beta_F(\log a) \geqq \kappa \varrho(\log a)\}.$$

Then there exist $\gamma > 0$, $q \geqq 0$ and for each $\kappa > 0$ a constant $C_\kappa > 0$ such that

$$(3) \qquad |f(a) - d_F(a)^{-1} f_F(a)| \leqq C_\kappa \Xi(a)^{1+\gamma\kappa} (1 + \sigma(a))^q \qquad (a \in A^+(F:\kappa)).$$

Finally, $F' \subseteqq F \subsetneq \Sigma$, we have the transitivity relation

$$(4) \qquad (f_F)_{F'} = f_{F'}.$$

THEOREM 2. *Let $G, f \neq 0$ be as in Theorem 1. Then the following statements are equivalent:* (i) *$f \in L^2(G) \otimes U$,* (ii) *$f_F = 0$ for all $F \subsetneq \Sigma$, and* (iii) *$f \in \mathscr{C}(G) \otimes U$. If these are satisfied, then $G = {}^\circ G$, and there exists $\gamma > 0$ such that $|afb| = O(\Xi^{1+\gamma})$ for all $a, b \in \mathfrak{G}$.*

THEOREM 3. *Let $\mathrm{rk}(G) = \mathrm{rk}(K)$. Let $\mathfrak{b} \subset \mathfrak{k}$ be a CSA and λ, a regular element of $\mathfrak{b}_c^*$ that is real valued on $(-1)^{1/2}\mathfrak{b}$. Then any $f \in C^\infty(G:\tau)$, which is tempered and satisfies the differential equations $zf = \chi_\lambda(z) f$ for all $z \in \mathfrak{Z}$, lies in $\mathscr{C}(G) \otimes U$. In particular, if G and Θ_λ are as in* §3 *$(\lambda \in \mathscr{L}')$, the Fourier components of Θ_λ are all in $\mathscr{C}(G)$.*

Our aim now is to sketch the main lines of arguments in the proofs of these theorems. We may assume $G = {}^\circ G$.

6.2. The differential equations on M_{1F}^+. Initial estimates. Fix F. For $m \in M_{1F}$ let $\gamma_F(m) = \|\mathrm{Ad}(m^{-1})_{\mathfrak{n}_F}\|$; $m \in M_{1F}^+ \Leftrightarrow \gamma_F(m) < 1$. Let $\mathscr{S}_F$ be the algebra of functions on M_{1F}^+ generated (without 1) by the derivatives of the matrix coefficients of the mappings

$$b_F(m \mapsto (\mathrm{Ad}(m^{-1}) - \mathrm{Ad}(\theta(m^{-1})))_{\mathfrak{n}_F}^{-1}) \quad \text{and} \quad c_F(m \mapsto \mathrm{Ad}(m^{-1})_{\mathfrak{n}_F} b_F(m)).$$

It is known [**14b**, §4] that for each $g \in \mathscr{S}_F$, there is $c = c(g) > 0$ and $r = r(g) \geqq 0$ such that

(1) $$|g(m)| \leqq c\gamma_F(m)\,(1-\gamma_F(m))^{-r} \qquad (m \in M_{1F}^+).$$

We note that $\mathfrak{G}$ is the direct sum of $\theta(\mathfrak{n}_F)\,\mathfrak{G}$ and $\mathfrak{M}_{1F}\mathfrak{K}$. Let $\nu_F:\mathfrak{G}\to\mathfrak{M}_{1F}\mathfrak{K}$ be the corresponding projection. It can be shown that

(2) $$\nu_F(z)=d_F^{-1}\circ\mu_F(z)\circ d_F \qquad (z\in\mathfrak{Z}).$$

The transfer of the differential equations satisfied by members of $\mathscr{A}(G:\tau)$, from G to M_{1F}^+, is based on the following [**14b**, §4]:

LEMMA 1. *Let $b\in\mathfrak{G}$. Then there exist $\eta_i\in\mathfrak{M}_{1F}$, ξ_i, $\zeta_i\in\mathfrak{K}$, and $g_i\in\mathscr{S}_F$ $(1\leqq i\leqq q)$ such that, for all $g\in C^\infty(G:\tau)$ and $m\in M_{1F}^+$,*

(3) $$g(m;b)=g(m;\nu_F(b))+\sum_{1\leqq i\leqq q} g_i(m)\,\tau_1(\xi_i)\,g(m;\eta_i)\,\tau_2(\zeta_i).$$

Let $f\in\mathscr{A}(G:\tau)$ be nonzero and $\mathfrak{Z}_f=\{z:z\in\mathfrak{Z},\ zf=0\}$. Define $\mathfrak{Z}_{f,F}=\mathfrak{Z}_F\mu_F[\mathfrak{Z}_f]$; then $l=\dim(\mathfrak{Z}_F/\mathfrak{Z}_{f,F})<\infty$. We select $u_1=1, u_2,\dots,u_l$ in $\mathfrak{Z}_F$ to be linearly independent modulo $\mathfrak{Z}_{f,F}$. Clearly there is a unique $l\times l$ matrix representation $\Gamma:\xi\mapsto\Gamma(\xi)=(c_{ij}(\xi))$ of $\mathfrak{Z}_F$ such that, for each $\xi\in\mathfrak{Z}_F$ and $1\leqq j\leqq l$,

(4) $$\xi u_j=\sum_{1\leqq i\leqq l} c_{ji}(\xi)\,u_i+\zeta_{j,\xi} \qquad (\zeta_{j,\xi}\in\mathfrak{Z}_{f,F}).$$

We define $\hat{U}=U\otimes\boldsymbol{C}^l$, $\hat{\tau}=\tau\otimes 1$, and choose an orthonormal basis $\{e_1,\dots,e_l\}$ for $\boldsymbol{C}^l$. For $m\in M_{1F}$, $\xi\in\mathfrak{Z}_F$, put

(5) $$\Phi(m)=\sum_{1\leqq j\leqq l} f(m;u_j\circ d_F)\otimes e_j,$$
$$\Psi(m:\xi)=\sum_{1\leqq j\leqq l} f(m;\zeta_{j,\xi}\circ d_F)\otimes e_j.$$

We then have the differential equations

(6) $$\Phi(m;\xi)=(1\otimes\Gamma(\xi))\,\Phi(m)+\Psi(m:\xi).$$

It is convenient to rewrite these in the following form.

LEMMA 2. *Let $\Sigma=\{\lambda_1,\dots,\lambda_p\}$, $F=\{\lambda_{d+1},\dots,\lambda_p\}$ and let $\{H_1,\dots,H_p\}$ be the basis*[16] *of $\mathfrak{a}$ dual to $\{\lambda_1,\dots,\lambda_p\}$. For $\boldsymbol{t}=(t_1,\dots,t_d)\in\boldsymbol{R}^d$, put*

$$a(\boldsymbol{t})=\exp(t_1H_1+\cdots+t_dH_d).$$

Then, for all $m\in M_{1F}^+$, $\eta\in\mathfrak{M}_{1F}$, $1\leqq j\leqq d$,

(7) $$\frac{\partial}{\partial t_j}\Phi(ma(\boldsymbol{t});\eta)=(1\otimes\Gamma(H_j))\,\Phi(ma(\boldsymbol{t});\eta)+\Psi(ma(\boldsymbol{t});\eta:H_j).$$

---

[16] Note that $H_j\in\mathfrak{Z}_F$ for $1\leqq j\leqq d$, and they span $\mathfrak{a}_F$.

For $t=(t_1,\ldots,t_d)$, let $|t|=(t_1^2+\cdots+t_d^2)^{1/2}$, $\min(t)=\min(t_1,\ldots,t_d)$. Define $R_+^d=\{t:t\in R^d, \min(t)>0\}$.

LEMMA 3. *Let $\Xi_F=\Xi_{M_{1F}}$. Then for each $\eta\in\mathfrak{M}_{1F}$, $\xi\in\mathfrak{Z}_F$ we can find $B=B_{\xi,\eta}>0$ and $q=q_{\xi,\eta}\geqq 0$ such that for all $m\in M_{1F}^+$, $t\in R_+^d$,*

$$(8)\qquad \begin{aligned}|\Phi(ma(t);\eta)|&\leqq B\Xi_F(m)(1+\sigma(m))^q(1+|t|)^q,\\ |\Psi(ma(t);\eta:\xi)|&\leqq B\Xi_F(m)(1+\sigma(m))^q\gamma_F(m)(1-\gamma_F(m))^{-q}(1+|t|)^q e^{-\min(t)}.\end{aligned}$$

For proving these we need to use the following consequence of (4.1.5): there exist $c_0>0$, $r_0\geqq 0$ such that

$$(9)\qquad d_F(m)\,\Xi(m)\leqq c_0\Xi_F(m)(1+\sigma(m))^{r_0}\qquad (m\in M_{1F}^+).$$

In addition, while deriving the estimates for Ψ we use: (i) the following inequality which follows from (1)–(3): given $\zeta\in\mathfrak{M}_{1F}$ and $z\in\mathfrak{Z}_f$, we can find $c_1>0$, $r_1\geqq 0$ such that, for all $m\in M_{1F}^+$,

$$|f(m;\zeta\mu_F(z)\circ d_F)|\leqq c_1 d_F(m)\,\Xi(m)(1+\sigma(m))^{r_1}\gamma_F(m)(1-\gamma_F(m))^{-r_1};$$

and (ii) the inequality $\gamma_F(ma(t))\leqq\gamma_F(m)\exp\{-\min(t)\}$ for all $m\in M_{1F}^+$, $t\in R_+^d$.

6.3. On some differential equations of first order. We shall now describe the technique which enables us to determine the asymptotic behaviour of Φ from the first order differential equations (6.2.7) and the estimates (6.2.8).

Let W be a Hilbert space of dimension $n<\infty$; $\Gamma_1,\ldots,\Gamma_d$, mutually commuting endomorphisms of W. For $1\leqq j\leqq d$ and $\mu\in C$, $W_{j,\mu}=\{w:w\in W, (\Gamma_j-\mu 1)^s w=0$ for some $s\geqq 0\}$. We put

$$(1)\qquad {}^\circ W=\bigcap_{1\leqq j\leqq d}\Big(\sum_{\mu:\,\mathrm{Re}\,\mu=0}W_{j,\mu}\Big),\qquad {}^\circ\Gamma_j=\Gamma_j\,|\,{}^\circ W;$$

of course, ${}^\circ W$ is invariant under all the Γ_j and the ${}^\circ\Gamma_j$ have only pure imaginary eigenvalues. We now consider functions F and G_j $(1\leqq j\leqq d)$, defined and C^1 in a neighborhood of $\mathrm{Cl}(R_+^d)$ with values in W, and having the following properties: (i) there exist $c>0$, $\beta>0$, $r\geqq 0$ such that, for all $t\in R_+^d$, $1\leqq j\leqq d$,

$$(2)\qquad \begin{aligned}|F(t)|\leqq c(1+|t|)^r,\qquad |G_j(t)|&\leqq c(1+|t|)^r e^{-\beta\min(t)},\\ (\partial F/\partial t_j)(t)&=\Gamma_j F(t)+G_j(t).\end{aligned}$$

LEMMA 1. *Let the notation be as above. Then there is a unique $w\in{}^\circ W$ such that, with $F_\infty(t)=\exp(t_1{}^\circ\Gamma_1+\cdots+t_d{}^\circ\Gamma_d)\,w$, $|F(t\tau)-F_\infty(t\tau)|\to 0$ as $t\to+\infty$, for each $\tau\in R_+^d$. Moreover, there are constants $C>0$, $\alpha>0$, depending only on r, β, n and the Γ_j such that, for all $t\in R_+^d$,*

(3) $$|F(t)-F_\infty(t)| \leqq Cc(1+|t|)^{r+2n} \exp\{-\alpha \min(t)\},$$
$$|F_\infty(t)| \leqq Cc(1+|t|)^n.$$

Any endomorphism T of W can be uniquely written as $T'+T''$, where T', $T'' \in \boldsymbol{C}[T]$, T' is semisimple, and the spectrum of T' (resp. T'') is real (resp. pure imaginary); moreover (see [**5e**, Lemma 60]) there is a constant $k(n)>0$ independent of T such that $\|e^{T''}\| \leqq k(n)(1+\|T''\|)^{n-1}$. We may thus replace Γ_j by Γ_j' and come down to the case when $\Gamma_j = c_j 1$ for all j, with $\boldsymbol{c}=(c_1, \ldots, c_d) \in \boldsymbol{R}^d$. If some $c_i>0$, we select $\gamma_1, \ldots, \gamma_d > 0$ with[17] $(\gamma, c)>0$ and find that, for all $t \in \boldsymbol{R}^d_+$,

$$F(t) = -\sum_j \gamma_j \int_0^\infty \exp\{-x(\gamma, c)\} G_j(t+x\gamma)\, dx.$$

If $\boldsymbol{c} \neq \boldsymbol{0}$ but $c_i \leqq 0$ for all i, we find that, for all $t \in \boldsymbol{R}^d_+$,

$$F(t) = \exp\{(c, t)\} \left\{ F(\boldsymbol{0}) + \sum_j t_j \int_0^1 \exp\{-x(c, t)\}\, G_j(xt)\, dx \right\}.$$

Finally, suppose $\boldsymbol{c}=\boldsymbol{0}$. Then $F(\tau, \ldots, \tau) \to$ a limit w as $\tau \to +\infty$, and for all $\tau>0$, writing $\boldsymbol{\tau}=(\tau, \ldots, \tau)$, $F(\boldsymbol{\tau}) = w - \sum_j \int_\tau^\infty G_j(x, \ldots, x)\, dx$. We then have, for all $t \in \boldsymbol{R}^d_+$, with $\tau = \min(t)$,

$$F(t) - w = (F(\boldsymbol{\tau}) - w) + \sum_j (t_j - \tau) \int_0^1 G_j(\boldsymbol{\tau} + x(t-\boldsymbol{\tau}))\, dx.$$

The required estimates follow from these formulae.

6.4. Proofs of the theorems. We now apply the results of §6.3 to (6.2.7) and (6.2.8) to deduce the existence of a function $\Phi_F \in C^\infty(M_{1F}: \hat{\tau}_F)$ with values in ${}^\circ\hat{U}$, having the following properties:

(i) $\Phi_F(m \exp H) = e^{1 \otimes \Gamma(H)} \Phi_F(m)$ for all $m \in M_{1F}$, $H \in \mathfrak{a}_F$,

(ii) $\Phi_F(m; \xi) = (1 \otimes \Gamma(\xi))\, \Phi_F(m)$ for all $m \in M_{1F}$, $\xi \in \mathfrak{Z}_F$, and

(iii) there are constants $\alpha>0$, $B_1>0$, $q_1 \geqq 0$ such that

(1) $$|\Phi_F(m)| \leqq B_1 \Xi_F(m)(1+\sigma(m))^{q_1} \qquad (m \in M_{1F}),$$

(2) $$\begin{aligned} &|\Phi(m \exp H) - e^{1 \otimes \Gamma(H)} \Phi_F(m)| \\ &\quad \leqq B_1 \Xi_F(m)(1+\sigma(m))^{q_1}(1-\gamma_F(m))^{-q_1} \\ &\quad \cdot (1+\|H\|)^{q_1} \exp\{-\alpha \beta_F(H)\} \qquad (m \in M_{1F}^+, H \in \mathfrak{a}_F^+). \end{aligned}$$

[17] $(\boldsymbol{a}, \boldsymbol{b}) = \sum_j a_j b_j$ for $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}^d$.

Actually, the existence of Φ_F with these properties on M_{1F}^+ is more or less immediate; the extension of Φ_F to M_{1F} so that (i), (ii) and (1) of (iii) above are valid for all $m \in M_{1F}$ is made possible by the following result (cf. [**6h**, Lemma 54]): Given $\bar{H} \in \mathfrak{a}_F^+$, there exists $c(\bar{H})>0$ such that $m \exp t\bar{H} \in M_{1F}^+$ for all $m \in M_{1F}$ and $t \geqq c(\bar{H}) \cdot \sigma(m)$.

Write $\Phi_F = \sum_{1 \leqq j \leqq l} \Phi_{F,j} \otimes e_j$ and define $f_F = \Phi_{F,1}$. Then $f_F \in \mathscr{A}(M_{1F}, \tau_F)$, and it is clear from (2) that (6.1.1) is valid. If $z \in \mathfrak{Z}_f$, $\Gamma(\mu_F(z))=0$, and so $\mu_F(z)\, f_F=0$. For any $a \in A^+$ let us write $\log a = H_1 + H_2$ where $H_1 \in \mathfrak{a}_F$ and $\lambda(H_2)=0$ for all $\lambda \in \Sigma \setminus F$. If we take $m=\exp(H_2+\frac{1}{2}H_1)$, $H=\frac{1}{2}H_1$ in (2), and observe that for suitable constants $c'>0$, $r' \geqq 0$, $d_F^{-1}\Xi_F \leqq c'\Xi(1+\sigma)^{r'}$ on A^+ (cf. (4.1.5)), we obtain the following estimate: There exist $B_2>0$, $q_2 \geqq 0$, $\gamma>0$ such that for all $a \in \mathrm{Cl}(A^+)$ with $\beta_F(\log a) \geqq 1$,

$$(3) \qquad |f(a)-(d_F^{-1}f_F)(a)| \leqq B_2(1+\sigma(a))^{q_2} \exp\{-\varrho(\log a)-\gamma\beta_F(\log a)\}.$$

From this we get (6.1.3) and hence (6.1.4), without difficulty.

Now we can find $\kappa_0>0$ such that $\mathrm{Cl}(A^+) \subseteq \bigcup_{F \subsetneqq \Sigma} A^+(F:\kappa_0)$. So, if $f_F=0$ for all F, (6.1.3) implies that $|f|=O(\Xi^{1+\gamma\kappa_0})$, thus proving the implication (ii)$\Rightarrow$(iii) of Theorem 6.1.2. Suppose now that $f \in L^2(G) \otimes U$. Let J be as in (4.1.6), and J_F, the corresponding function for M_{1F}. Clearly, given any $y>0$, we can find $c(y)>0$ such that for all $a \in A^+$ with $\beta_F(\log a) \geqq y$, $d_F(a)^2 J_F(a) \leqq c(y)\, J(a)$. So, writing $A_y^+(F:\kappa)=\{a : a \in A^+, \beta_F(\log a) \geqq \max(y, \kappa\varrho(\log a))\}$, we find from (6.1.3) that

$$(4) \qquad \int_{A^+{}_y(F:\kappa)} J_F(a)\, |f_F(a)|^2\, da < \infty.$$

If we remember that $J_F(aa')=J_F(a')$ for all $a' \in A_F$ and that $a' \mapsto f_F(ma')$ is a tempered $\mathfrak{A}_F$-finite function on A_F, we can deduce from (4) that $f_F=0$.

We now consider Theorem 3. We extend $\mathfrak{a}$ to a θ-stable CSA $\mathfrak{l}$, and assume that $zf=\chi_\Lambda(z)f$, $\forall z \in \mathfrak{Z}$, $\Lambda \in \mathfrak{l}_c^*$ being regular and real-valued on $(-1)^{1/2}(\mathfrak{l}\cap\mathfrak{l})+(\mathfrak{l}\cap\mathfrak{p})$. We need the following lemma.

LEMMA 1. *Given $F \subsetneqq \Sigma$, there exists a unique ${}^\circ f_F \in \mathscr{A}(M_F : \tau_F)$ such that $f_F(ma) = {}^\circ f_F(m)$ for all $m \in M_F$, $a \in A_F$; moreover, $f_F=0$ unless $s\Lambda \mid \mathfrak{a}_F=0$ for some $s \in W_{L_c}$.*

To prove the lemma, one first uses the differential equation $\mu_F(z)\, f_F=\chi_\Lambda(z)\, f_F$ $(z \in \mathfrak{Z})$ and the regularity of Λ (cf. also Lemma 6.5.6) to conclude the following: There exist unique functions $f_{F,j}$ on M_F such that $f_F(ma)=\sum_{1 \leqq j \leqq N} \exp\{\Lambda_j(\log a)\} \cdot f_{F,j}(m)(m \in M_F, a \in A_F)$, $\Lambda_1, \ldots, \Lambda_N$ being all the distinct ones among the restrictions $s\Lambda \mid \mathfrak{a}_F$ $(s \in W_{L_c})$. As the Λ_j are real, the tempered nature of $a \mapsto f_F(ma)$ implies that $f_{F,j}=0$ unless $\Lambda_j=0$. The assertions of the lemma follow from this.

Suppose now that the conditions of Theorem 3 are satisfied but $f \notin \mathscr{C}(G) \otimes U$.

We select $F \subsetneq \Sigma$ of the smallest cardinality such that $f_F \neq 0$. Then $({}^\circ f_F)_{F'} = 0$ for all $F' \subsetneq F$, by (6.1.4). So ${}^\circ f_F \in \mathscr{C}(M_F) \otimes U$, implying that $\mathrm{rk}(M_F) = \mathrm{rk}(K_F)$ (§5.7). This means that, for some θ-stable CSA $\mathfrak{h}$, $\mathfrak{h} \cap \mathfrak{p} = \mathfrak{a}_F$. But $\mathfrak{h}$ is not conjugate to $\mathfrak{b}$ and so $(\mathfrak{g}, \mathfrak{h})$ has a real root α. This implies that $H_\beta \in \mathfrak{a}_F$ for some root β of $(\mathfrak{g}, \mathfrak{l})$. But then, by Lemma 1, $\langle s\Lambda, \beta \rangle = 0$ for some $s \in W_{L_c}$, contradicting the regularity of Λ.

6.5. Estimates uniform over the discrete spectrum. We have not made full use of the techniques of §§6.2 and 6.3 in proving the main theorems of this section. It turns out that one can obtain estimates describing the asymptotic behaviour of an eigenfunction that are actually uniform over the spectrum as well as over the set of parameters of the representations of K according to which the eigenfunction transforms when subjected to translations from K. Obviously, such estimates will play an important role in harmonic analysis on G (**[5e]**, **[5f]**, **[14a]**). In this section we shall illustrate what is involved by discussing the case when the eigenfunctions come from the discrete spectrum.

Let G be as in §3. $\mathfrak{l}$ is a θ-stable CSA containing $\mathfrak{a}$; $\mathscr{L}'(\mathfrak{l})$ the set of integral regular elements in $\mathfrak{l}_c^*$. Given $\Lambda \in \mathscr{L}'(\mathfrak{l})$ and τ, U as before, we define

$$\mathscr{A}(G:\tau:\Lambda) = \{f : f \in \mathscr{A}(G:\tau), zf = X_\Lambda(z) f, \forall z \in \mathfrak{Z}\}. \tag{1}$$

Then $\mathscr{A}(G:\tau:\Lambda) \subseteq \mathscr{C}(G) \otimes U$. We define Ω as in §1, select a norm $\|\cdot\|$ in $\mathfrak{l}_c^*$, and put

$$|\tau| = (1 + \|\tau_1(\Omega)\|)(1 + \|\tau_2(\Omega)\|), \qquad |\tau, \Lambda| = |\tau|(1 + \|\Lambda\|). \tag{2}$$

For any measurable function $\varphi: G \to U$, $\|\varphi\|_2$ is the L^2-norm of the function $x \to |\varphi(x)|$ whenever this is finite. Our main result in this section is then the following:

THEOREM 1. *We can find a constant $\alpha > 0$, and, corresponding to any $a, b \in \mathfrak{G}$, constants $C = C_{a,b} > 0$, $r = r_{a,b} \geqq 0$ such that, for all $\Lambda \in \mathscr{L}'(\mathfrak{l})$, all τ, and all $f \in \mathscr{A}(G:\tau:\Lambda)$,*

$$|(afb)(x)| \leqq C|\tau, \Lambda|^r \|f\|_2 \Xi(x)^{1+\alpha} \qquad (x \in G). \tag{3}$$

We remark that it is enough to prove this theorem for $a = b = 1$; the general case can then be deduced through an elementary device. Further, there exists $\kappa > 0$ such that

$$\mathrm{Cl}(A^+) \subseteq \bigcup_{\lambda \in \Sigma} A^+(\Sigma \backslash \{\lambda\} : \kappa).$$

It is therefore sufficient to establish the following:

LEMMA 2. *Fix $\lambda \in \Sigma$ and let $F = \Sigma \backslash \{\lambda\}$. Then there exist $\alpha > 0$, $C > 0$, $q \geqq 0$ such* *that for all* Λ, τ, $\kappa > 0$, *as above*, $f \in \mathscr{A}(G:\tau:\Lambda)$,

$$|f(a)| \leqq C|\tau, \Lambda|^q \, \Xi(a)^{1+\alpha\kappa} \|f\|_2 \qquad (a \in A^+(F:\kappa)). \tag{4}$$

For the rest of this section we fix F as above, and $H \in \mathfrak{a}_F^+$; note that $\dim(\mathfrak{a}_F) = 1$. First we have the following a priori estimates:

LEMMA 3. *Let* $u, v \in \mathfrak{G}$. *Then there exist* $C = C_{u,v} > 0$ *and* $s = s_{u,v} \geqq 0$ *such that, for all* Λ, τ *as above and* $f \in \mathscr{A}(G:\tau:\Lambda)$,

$$|(ufv)(x)| \leqq C|\tau, \Lambda|^s \, \Xi(x) \|f\|_2 \qquad (x \in G). \tag{5}$$

Given $a, b \in \mathfrak{G}$, we first prove the existence of a constant $C_1 = C_{1,a,b} > 0$ such that $\|afb\|_2 \leqq C_1 |\tau, \Lambda|^d \, \|f\|_2$ for all τ, Λ, f as above, where $d = \deg(a) + \deg(b)$ (see [**14b**, Lemma 5.5]); (5) now follows from Theorem 4.1.3.

We next introduce Φ and its differential equations. We select $v_1 = 1, v_2, \ldots, v_l \in \mathfrak{Z}_F$ such that $\mathfrak{Z}_F$ is the direct sum of $\mu_F[\mathfrak{Z}]\, v_i$ $(1 \leqq i \leqq l)$. Define $\hat{U} = U \otimes \boldsymbol{C}^l$ and select an orthonormal basis $\{e_i\}$ for $\boldsymbol{C}^l$. If $v \in \mathfrak{Z}_F$, $vv_j = \sum_{1 \leqq i \leqq l} \mu_F(z_{v:ij})\, v_i$ $(1 \leqq j \leqq l)$ for unique $z_{v:ij} \in \mathfrak{Z}$ and we write $\Gamma(\Lambda:v)$ for the $l \times l$ matrix whose ijth element is $\mu_{\mathfrak{g}/\mathfrak{t}}(z_{v:ji})(\Lambda)$. $\Gamma(\Lambda:\cdot)$ is a representation of $\mathfrak{Z}_F$ in $\boldsymbol{C}^r$. Given $f \in \mathscr{A}(G:\tau:\Lambda)$ we write

$$\Phi(m) = \sum_{1 \leqq j \leqq l} f(m; v_j) \otimes e_j \qquad (m \in M_{1F}). \tag{6}$$

Proceeding as in §6.2, but taking into account the variability of τ, we get the following result [**14b**, Lemmas 5.2 and 5.4]:

LEMMA 4. *Let* $v \in \mathfrak{Z}_F$. *Then for each* τ *we can find a differential operator* D_v^τ *acting on* $C^\infty(M_{1F}^+ : \hat{U})$ *such that*

(i) *for all* τ, Λ *as above and* $f \in \mathscr{A}(G:\tau:\Lambda)$,

$$\Phi(m; v) = (1 \otimes \Gamma(\Lambda:v))\, \Phi(m; v) + \Phi(m; D_v^\tau) \qquad (m \in M_{1F}^+); \tag{7}$$

(ii) *there exist* $r \geqq 0$, $\omega_k \in \mathfrak{M}_{1F}$ $(1 \leqq k \leqq k_0)$ *such that, for all* τ *as above and all* $g \in C^\infty(M_{1F}^+ : \hat{U})$,

$$|g(m; D_v^\tau)| \leqq \gamma_F(m)\,(1 - \gamma_F(m))^{-r}\, |\tau|^r \sum_{1 \leqq k \leqq k_0} |f(m; \omega_k)|. \tag{8}$$

We now have the differential equations

$$\frac{d}{dt}\Phi(m \exp tH) = (1 \otimes \Gamma(H))\, \Phi(m \exp tH) + \Phi(m \exp tH; D_H^\tau) \tag{9}$$

valid for all $m \in M_{1F}^+$, $t \geqq 0$. Furthermore, as our eigenfunctions f are in $\mathscr{C}(G) \otimes U$,

$$\lim_{t\to+\infty} \Phi(m \exp tH) = 0. \tag{10}$$

We are therefore in a position to proceed as in §6.4. However, Lemma 6.3.1 cannot be used as it is, because the estimates given by it are not uniform over $\Gamma_1, \ldots, \Gamma_d$. We therefore use the following variant:

LEMMA 5. *Let W be as in §6.3, and Γ a semisimple endomorphism of W whose spectrum $S=S(\Gamma)$ is real. Let E_c $(c\in S)$ be the spectral projections and define*

$$\nu(\Gamma)=\max_{c\in S} \|E_c\|, \qquad \sigma(\Gamma)=\min_{c\in S\setminus\{0\}} |c|. \tag{11}$$

Let F, H be functions of class C^1 on $[0,\infty)$ with values in W such that (i) *$dF/dt=\Gamma F+H$ on $[0,\infty)$,* (ii) *there exist $C>0$, $r\geqq 0$, $\beta>0$ such that $|F(t)|\leqq C(1+t)^r$, $|H(t)|\leqq C(1+t)^r e^{-\beta t}$ for all $t\geqq 0$, and* (iii) *the limit $\lim_{t\to+\infty} F(t)$ exists and is 0. Then, with $[S]$ denoting the number of elements of S and $A(r,\beta)>0$ a constant depending on r and β but not on Γ or C,*[18] *we have*

$$|F(t)|\leqq A(r,\beta)\, C\nu(\Gamma)\, [S(\Gamma)]\, (1+t)^r \exp\{-\min(\beta,\sigma(\Gamma))\, t\} \qquad (t\geqq 0). \tag{12}$$

On the other hand, the spectral structure of the matrices $\Gamma(\Lambda:v)$ is known in great detail and one has the following lemma ([**5e**, §3], [**5f**, Lemma 19], [**14b**, Lemmas 5.1 and 7.2]):

LEMMA 6. *For each $v\in\mathfrak{Z}_F$, $\Lambda\in\mathscr{L}'(\mathfrak{l})$, $\Gamma(\Lambda:v)$ is semisimple and its eigenvalues are $\mu_{\mathfrak{m}_{1F}/\mathfrak{l}}(v)$ $(s\Lambda)$ $(s\in W_{L_c})$. In particular, the eigenvalues λ of $\Gamma(\Lambda:H)$ are real, and there exists $\alpha>0$ such that $|\lambda|\geqq\alpha$ for all nonzero λ and all $\Lambda\in\mathscr{L}'(\mathfrak{l})$. Moreover, we can choose a basis $e_j(\Lambda)$ for $\mathbf{C}^l$ $(1\leqq j\leqq l$, $\Lambda\in\mathscr{L}'(\mathfrak{l}))$ such that* (i) *all the $\Gamma(\Lambda:v)$ are diagonal in this basis, and* (ii) *if $E_j(\Lambda)$ are the projections $\mathbf{C}^l\to\mathbf{C}\cdot e_j(\Lambda)$, then there exist $C_0>0$, $r_0\geqq 0$ such that, for all $\Lambda\in\mathscr{L}'(\mathfrak{l})$,*

$$\sum_{1\leqq j\leqq l} \|E_j(\Lambda)\|\leqq C_0(1+\|\Lambda\|)^{r_0}. \tag{13}$$

Lemma 2 and thence Theorem 1 follow from these estimates more or less in the same way as in §6.4. For details see [**14b**].

The perturbation method is central in the entire theory of asymptotic behaviour of eigenfunctions, and the results obtained through its application go far beyond what we have indicated above. As further examples we mention the theorems that suitably formed "wave packets" (of eigenfunctions) over the spectrum belong to $\mathscr{C}(G)$ and even $\mathscr{C}^1(G)$ (cf. [**6i**], [**14a**]), as well as the results on the theory of integrable eigenfunctions (cf. [**14b**]).

[18] We can take $A(r,\beta)=\max(2, \int_0^\infty (1+u)^r e^{-\beta u}\, du)$.

7. The discrete series

We shall now describe briefly how the results of the preceding chapters lead to the determination of the discrete series. Exploiting the fact that the K-finite matrix coefficients of representations of the discrete series are cusp forms, one uses Theorem 5.7.1 to reduce their harmonic analysis to that on the compact CSG; the procedure is of course similar to that used by Weyl for determining the characters of compact groups.

7.1. The discrete series and the discrete part of the Plancherel formula. For simplicity we restrict ourselves to the case when G is a connected real form of a complex simply connected semisimple group G_c. We choose once and for all a Haar measure dx and fix it throughout this section. We write $\mathscr{E}_2(G)$ for the discrete series of G. For $\omega\in\mathscr{E}_2(G)$, $d(\omega)>0$ is its formal degree so that, for any $\pi\in\omega$ and unit vectors ϕ, ψ in the space of π,

(1) $$\int_G |(\pi(x)\varphi, \psi)|^2\, dx = d(\omega)^{-1}.$$

$L^2_\omega(G)$ is the closed linear span of the matrix coefficients of ω; it coincides with the closed linear span of all subspaces of $L^2(G)$ that are irreducibly invariant under the right regular representation r and define a subrepresentation in ω. ${}^\circ L^2(G)$ is the (orthogonal) direct sum of the $L^2_\omega(G)$. ${}^\circ E$ and E_ω are the orthogonal projections of $L^2(G)$ on ${}^\circ L^2(G)$ and $L^2_\omega(G)$ respectively. ω^* is the class contragredient to ω.

THEOREM 1. *G has a discrete series if and only if* $\mathrm{rk}(G)=\mathrm{rk}(K)$, *i.e., G has a compact CSG.*

Assume now that $\mathrm{rk}(G)=\mathrm{rk}(K)$ and use the notation of §3. In particular, let P be a positive system of roots of $(\mathfrak{g}, \mathfrak{b})$; $\delta=\frac{1}{2}\sum_{\alpha\in P}\alpha$, $\Delta=\xi_{-\delta}\prod_{\alpha\in P}(\xi_\alpha-1)$, $\varpi=\prod_{\alpha\in P}H_\alpha$, $\varepsilon(\lambda)=\operatorname{sign}\varpi(\lambda)\ (\lambda\in\mathscr{L}')$, $q=\frac{1}{2}\dim(G/K)$.

THEOREM 2. *For each $\lambda\in\mathscr{L}'$, there exists $\omega[\lambda]\in\mathscr{E}_2(G)$ such that $\Theta_{\omega[\lambda]}=(-1)^q\varepsilon(\lambda)\Theta_\lambda$, and every $\omega\in\mathscr{E}_2(G)$ is of the form $\omega[\lambda]$ for some $\lambda\in\mathscr{L}'$. $\omega[\lambda_1]=\omega[\lambda_2]$ if and only if λ_1 and λ_2 are in the same W_B-orbit. Moreover, there exists a constant $c(G)>0$ such that $d(\omega[\lambda])=c(G)\,[W_B]\,|\varpi(\lambda)|$ for all $\lambda\in\mathscr{L}'$. Finally, $\omega[-\lambda]=\omega[\lambda]^*$.*

Harish-Chandra has determined the value of $c(G)$ explicitly when the Haar measure dx is normalized in a canonical way (cf. [**6i**, pp. 537 and 540]).

For $\lambda\in\mathscr{L}'$ and $f\in\mathscr{C}(G)$, let

(2) $$f_\lambda(x)=(-1)^q c(G)\,[W_B]\,\varpi(-\lambda)\,\Theta_{-\lambda}(r(x)f)\qquad(x\in G).$$

THEOREM 3. *For any* $f \in \mathscr{C}(G)$ *and* $\lambda \in \mathscr{L}'$, $f_\lambda = E_{\omega[\lambda]} f$ *and lies in* $\mathscr{C}(G)$. *Moreover, the series*

(3) $$\sum_{\lambda \in \mathscr{L}'/W_B} f_\lambda$$

converges absolutely[19] *in* $\mathscr{C}(G)$, *and its sum* $^\circ f$ *is precisely* $^\circ Ef$. *Finally,* $^\circ\mathscr{C}(G) = \mathscr{C}(G) \cap {}^\circ L^2(G)$, *and* $f \to {}^\circ f$ *is a continuous map of* $\mathscr{C}(G)$ *onto* $^\circ\mathscr{C}(G)$.

7.2. Theorems 1 and 2. Standard arguments from representation theory [**6h**, Lemma 77] show that $\mathscr{E}_2(G) \neq \emptyset$ if and only if there are nonzero K-finite eigenfunctions for $\mathfrak{Z}$ in $L^2(G)$. Theorem 1 is then immediate from Theorems 5.7.3, 6.1.2 and 6.1.3.

We now come to Theorem 2 (for complete details, see [**6h**, §§40, 41]). Let $\mathrm{rk}(G) = \mathrm{rk}(K)$. For any $g \in C^\infty(B)$ let $\hat{g}: \lambda \mapsto \int g\xi_\lambda \, db$ $(\lambda \in \mathscr{L})$ be its Fourier transform ($\int_B db = 1$). Define

(1) $$F_f(b) = \Delta(b) \int_G f(xbx^{-1})\, dx \qquad (b \in B',\ f \in \mathscr{C}(G)).$$

Then Theorems 5.5.1 and 5.7.1 imply easily that, for all cusp forms f and all tempered invariant eigendistributions Θ,

(2) $$\Theta(f) = (-1)^m [W_B]^{-1} \int_B \Phi F_f \, db \qquad (m = \tfrac{1}{2}\dim(G/B)),$$

Φ being the analytic function on B that extends $\Delta(\Theta \mid B')$; in particular, for all $\lambda \in \mathfrak{L}'$,

(3) $$\Theta_\lambda(f) = (-1)^m \hat{F}_f(\lambda).$$

If we now take Fourier transforms in the relation (5.6.2) and remember that $F_f \in C^\infty(B)$, we obtain the following result: there is a constant $c(G) > 0$ ($c(G)$ is the constant c of (5.6.2)) such that

(4) $$f(x) = (-1)^q c(G) \sum_{\lambda \in \mathscr{L}'} \varpi(\lambda)\, \Theta_\lambda(r(x)\, f) \qquad (x \in G,\ f \in {}^\circ\mathscr{C}(G)).$$

It is clear from (4) that the harmonic analysis of the cusp forms is completely controlled by the distributions Θ_λ.

We shall now indicate how the transition from (4) to Theorem 2 is carried out. Fix a homomorphism χ of $\mathfrak{Z}$ into $\mathbf{C}$ and let $\mathscr{E}_{2,\chi}$ be the set of all $\omega \in \mathscr{E}_2(G)$ with $\chi_\omega = \chi$. If $\omega \in \mathscr{E}_{2,\chi}$ and g is a matrix coefficient of ω, then $\Theta_\mu(r(x)\, g) = 0$ unless

[19] This means that for any continuous seminorm ν on $\mathscr{C}(G)$, $\sum_\lambda \nu(f_\lambda) < \infty$.

$\chi = \chi_{-\mu}$. So, taking $f = g$ in (4) we find that $\mathscr{E}_{2,\chi} = \emptyset$ unless $\chi = \chi_\lambda$ for some $\lambda \in \mathfrak{L}'$. For such a χ, we obtain from Corollary 5.5.3 and the results of §3 the following: Let $s_1 = 1, s_2, \ldots, s_r$ be a complete system of representatives for $W_B \backslash W_{B_c}$; let Φ_λ (resp. Φ_ω) be the analytic function on B extending $\Delta(\Theta_\lambda \mid B')$ (resp. $\Delta(\Theta_\omega \mid B')$); then, there is a unique $c_{\omega i} \in C(\omega \in \mathscr{E}_{2,\chi}, 1 \leqq i \leqq r)$ such that

$$\Theta_\omega = \sum_{1 \leqq i \leqq r} c_{\omega i} \Theta_{s_i \lambda}, \qquad \Phi_\omega = \sum_{1 \leqq i \leqq r} c_{\omega i} \Phi_{s_i \lambda}. \tag{5}$$

We now use the orthogonality relations satisfied by the matrix coefficients of the discrete series to obtain the following relations, valid for all $\omega \in \mathscr{E}_2(G)$:

$$\begin{aligned} F_f &= d(\omega)^{-1} f(1)\, \Phi_\omega \qquad (f \in \mathscr{C}(G) \cap L^2_\omega(G)), \\ \Theta_\omega(f) &= d(\omega)^{-1} f(1)\, \delta_{\omega'\omega^*} \quad (f \in \mathscr{C}(G) \cap L^2_{\omega'}(G)). \end{aligned} \tag{6}$$

It follows from these relations that the functions $[W_B]^{-1/2}\, \Phi_\omega$ $(\omega \in \mathscr{E}_{2,\chi})$ are orthonormal in $L^2(B)$. Moreover, they have the same span as the $\Phi_{s_i\lambda}$ $(1 \leqq i \leqq r)$. For, if this were not so, we could find a nonzero linear combination Θ of the $\Theta_{s_i\lambda}$ such that $\Theta(f) = 0$ for all K-finite eigenfunctions f for $\mathfrak{Z}$ in $\mathscr{C}(G)$; taking f to be an arbitrary Fourier component of Θ^{conj} we find that $\Theta = 0$. It follows at this stage that $\mathscr{E}_{2,\chi}$ has r elements and that the matrix $(c_{\omega i})$ $(\omega \in \mathscr{E}_{2,\chi}, 1 \leqq i \leqq r)$ is unitary.

One now argues that the $c_{\omega i}$ are integers. To see this, let $\omega \in \mathscr{E}_{2,\chi}$ and let $n(\mathfrak{d}) = [\omega : \mathfrak{d}]$, $\psi_\mathfrak{d}$ = character of $\mathfrak{d}$ $(\mathfrak{d} \in \mathscr{E}(K))$. Then $\sum_{\mathfrak{d} \in \mathscr{E}(K)} n(\mathfrak{d})\, \psi_\mathfrak{d}$ is a well-defined distribution on K, and one can show that it coincides on $K \cap G'$ with the distribution defined thereon by the function Θ_ω. It follows without difficulty from this that all the $c_{\omega i} \in \mathbf{Z}$. This completes the proof of Theorem 2 except for the formula for the formal degree and the sign factors in $\Theta_{\omega[\lambda]}$.

For determining the signs we argue as follows: We have $\Theta_{\omega[\lambda]} = \zeta(\lambda)\, \Theta_\lambda$ where ζ is a W_B-skew function on $\mathscr{L}'$ with values ± 1. Fix $\lambda \in \mathscr{L}'$ and use (4) with $x = 1$ and $f = g * \tilde{g}$ where $g \neq 0$ is a K-finite matrix coefficient of $\omega[\lambda]^*$. Then $f \in {}^\circ\mathscr{C}(G) \cap L^2_{\omega[\lambda]^*}(G)$, and

$$\|g\|^2 = (-1)^q\, c(G)\, \varpi(\lambda)\, \zeta(\lambda)\, \Theta_{\omega[\lambda]}(g * \tilde{g}).$$

This determines $\zeta(\lambda)$ and $d(\omega[\lambda])$, and yields the *discrete part* of the Plancherel formula:

$$\|f\|^2 = c(G)\, [W_B] \sum_{\lambda \in \mathscr{L}'/W_B} |\varpi(\lambda)| \Theta_{\omega[\lambda]}(f * \tilde{f}) \qquad (f \in {}^\circ\mathscr{C}(G)). \tag{7}$$

7.3. Outline of the proof of Theorem 3. For any $\lambda \in \mathscr{L}'$ and $f \in \mathscr{C}(G)$, the relation $f_\lambda = E_{\omega[\lambda]} f$ follows from a "real variables" argument. Also, $f \in \mathscr{C}(G)$ is a differentiable vector for both the left and right regular representations, as can be easily deduced from the estimates of §4.1. So ${}^\circ Ef$ and $E_\omega f$ are differentiable likewise. Consequently they are C^∞ functions all of whose derivatives lie in $L^2(G)$.

Write $\mathscr{L}^+$ for the set of all $\lambda \in \mathscr{L}'$ such that $\langle \lambda, \alpha \rangle > 0$ for all compact roots $\alpha \in P$.[20] For each $\lambda \in \mathscr{L}^+$ we select a Hilbert space $\mathfrak{H}_\lambda$, a $\pi_\lambda \in \omega[\lambda]$ acting in $\mathfrak{H}_\lambda$, and an orthonormal basis $\{e_{\lambda,i} : i \in N_\lambda\}$ for $\mathfrak{H}_\lambda$ such that each $e_{\lambda,i}$ belongs to a subspace irreducibly invariant under $\pi_\lambda[K]$. Let

$$a_{\lambda,i,j}(x) = d(\omega[\lambda])^{1/2} (\pi_\lambda(x)\, e_{\lambda,j}, e_{\lambda,i}) \qquad (x \in G).$$

Clearly, the $a_{\lambda,i,j}$ form an orthonormal basis of ${}^\circ L^2(G)$, and, for all $f \in \mathscr{C}(G)$,

$$(1) \qquad {}^\circ Ef = \sum_{\lambda \in \mathscr{L}^+} f_\lambda, \qquad f_\lambda = \sum_{i,j \in N_\lambda} f_{\lambda,i,j}, \qquad f_{\lambda,i,j} = (f, a_{\lambda,i,j})\, a_{\lambda,i,j}.$$

Let Ω be as in §§1 and 6.5, $z = \omega + \langle \delta \cdot \delta \rangle + 1$ where ω is the Casimir of G. Then $z a_{\lambda,i,j} = (1 + \|\lambda\|^2)\, a_{\lambda,i,j}$, $\Omega^r a_{\lambda,i,j} \Omega^r = c^r_{\lambda,i} c^r_{\lambda,j} a_{\lambda,i,j}$; moreover, the $c_{\lambda,i}$ are constants $\geqq 1$ and have the property that, for some $q \geqq 0$,

$$(2) \qquad c = \sup_{\lambda \in \mathscr{L}^+} \sum_{i \in N_\lambda} c_{\lambda,i}^{-q} < \infty.$$

We now use the uniform estimates of §6.5 to establish the following: there exist $\alpha > 0$, $C > 0$, $p \geqq 0$ such that for all $\lambda \in \mathscr{L}^+$, $i, j \in N_\lambda$, $x \in G$,

$$(3) \qquad |a_{\lambda,i,j}(x)| \leqq C[c_{\lambda,i} c_{\lambda,j} (1 + \|\lambda\|^2)]^p\, \Xi(x)^{1+\alpha}.$$

From (2) and (3) it is easy to deduce the following: There exist $C > 0$ and $m \geqq 0$ such that, for all $f \in \mathscr{C}(G)$,

$$(4) \qquad \sum_{\lambda \in \mathscr{L}^+} \sum_{i,j \in N_\lambda} \|\Xi^{-(1+\alpha)} f_{\lambda,i,j}\|_\infty \leqq C \|\Omega^m z^m f \Omega^m\|_2.$$

The estimate (4), together with those obtained by replacing f with ufv ($u, v \in \mathfrak{G}$), led to Theorem 3 (cf. [**6i**], [**14b**]).

The above discussion also shows that the topology of ${}^\circ\mathscr{C}(G)$ is precisely the one induced by the seminorms $f \mapsto \|ufv\|_2$ ($u, v \in \mathfrak{G}$). It is even possible to restrict ourselves only to the seminorms $f \mapsto \|\xi^r f \xi^s\|_2$ ($r, s \geqq 0$, $\xi = 1 - (X_1^2 + \cdots + X_n^2)$ where the X_i are an orthonormal basis of $\mathfrak{g}$); this however needs some more work which we do not go into here.

I wish to acknowledge my very great indebtedness to the many long conversations with Professor Harish-Chandra on problems of harmonic analysis on semisimple Lie groups. In addition I have profited a great deal from my discussions with friends and colleagues here as well as at other institutions, and I am grateful to all of them.

[20] $\mathscr{L}^+$ is a system of representatives for $\mathscr{L}'/W_B$.

References

1. V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. of Math. (2) **48** (1947), 568–640. MR **9**, 133.

2. L. Ehrenpreis and F. I. Mautner

a) *Some properties of the Fourier transform on semisimple Lie groups*. I, Ann. of Math. (2) **61** (1955), 406–439. MR **16**, 1017.

b) *Some properties of the Fourier transform on semisimple Lie groups*. II, Trans. Amer. Math. Soc. **84** (1957), 1–55. MR **18**, 745.

c) *Some properties of the Fourier transform on semisimple Lie groups*. III, Trans. Amer. Math. Soc. **90** (1959), 431–484. MR **21** #1545.

3. I. M. Gel'fand and M. I. Graev

a) *On a general method of decomposing the regular representation of a Lie group into irreducible representation*, Dokl. Akad. Nauk. SSSR **92** (1953), 221–224. (Russian) MR **15**, 601.

b) *Analogue of the Plancherel formula for the classical groups*, Trudy Moskov. Mat. Obšč. **4** (1955), 375–404; English Transl., Amer. Math. Soc. Transl. (2) **9** (1958), 123–154. MR, **17**, 173; **19**, 1181.

4. Harish-Chandra

a) *Representations of a semisimple Lie group on a Banach space*. I, Trans. Amer. Math. Soc. **75** (1953), 185–243. MR **15**, 100.

b) *Representations of semisimple Lie groups*. II, Trans. Amer. Math. Soc. **76** (1954), 26–65. MR **15**, 398.

c) *Representations of semisimple Lie groups*. III, Trans. Amer. Math. Soc. **76** (1954), 234–253. MR **16**, 11.

d) *Representations of semisimple Lie groups*. IV, Amer. J. Math. **77** (1955), 743–777. MR **17**, 282.

e) *Representations of semisimple Lie groups*. V, Amer. J. Math. **78** (1956), 1–41. MR **18**, 490.

f) *Representations of semisimple Lie groups*. VI. *Integrable and square-integrable representations*, Amer. J. Math. **78** (1956), 564–628. MR **18**, 490.

g) *Representations of semisimple Lie groups*, Proc. Internat. Congr. Math. (Amsterdam, 1954), vol. 1, Noordhoff, Groningen; North-Holland, Amsterdam, 1957, pp. 299–304. MR **20** #1926.

h) *The Plancherel formula for complex semisimple Lie groups*, Trans. Amer. Math. Soc. **76** (1954), 485–528. MR **16**, 111.

i) *The characters of semisimple Lie groups*, Trans. Amer. Math. Soc. **83** (1956), 98–163. MR **18**, 318.

5. Harish-Chandra

a) *Differential operators on a semisimple Lie algebra*, Amer. J. Math. **79** (1957), 87–120. MR **18**, 809.

b) *Fourier transforms on a semisimple Lie algebra*. I, Amer. J. Math. **79** (1957). MR **19**, 293.

c) *Fourier transforms on a semisimple Lie algebra*. II, Amer. J. Math. **79** (1957), 653–686. MR **20** #2396.

d) *A formula for semisimple Lie groups*, Amer. J. Math. **79** (1957), 733–760. MR **20** #2633.

e) *Spherical functions on a semisimple Lie group*. I, Amer. J. Math. **80** (1958), 241–310. MR **20** #925.

f) *Spherical functions on a semisimple Lie group*. II, Amer. J. Math. **80** (1958), 553–613. MR **21** #92.

6. Harish-Chandra

a) *Invariant distributions on Lie algebras*, Amer. J. Math. **86** (1964), 271–309. MR **28** #5144.

b) *Invariant differential operators and distributions on a semisimple Lie algebra*, Amer. J. Math. **86** (1964), 534–564. MR **31** #4862a.

c) *Some results on an invariant integral on a semisimple Lie algebra*, Ann. of Math. (2) **80** (1964), 551–593. MR **31** #4862b.

d) *Invariant eigendistributions on a semisimple Lie algebra*, Inst. Hautes Études Sci. Publ. Math. No. 27 (1965), 5–54. MR **31** #4862c.

e) *Invariant eigendistributions on a semisimple Lie group*, Trans. Amer. Math. Soc. **119** (1965), 457–508. MR **31** #4862d.

f) *Discrete series for semisimple Lie groups*. I, Acta Math. **113** (1965), 241–318. MR **36** #2744.

g) *Two theorems on semisimple Lie groups*, Ann. of Math. (2) **83** (1966), 74–128. MR **33** #2766.

h) *Discrete series for semisimple Lie groups*. II. *Explicit determination of the characters*, Acta Math. **116** (1966), 1–111. MR **36** #2745.

i) *Harmonic analysis on semisimple Lie groups*, Bull. Amer. Math. Soc. **76** (1970), 529–551. MR **41** #1933.

7. S. Helgason, *Differential geometry and symmetric spaces*, Pure and Appl. Math., vol. 12, Academic Press, New York, 1962. MR **26** #2986.

8. T. Hirai

a) *Invariant eigendistributions of Laplace operators on real semisimple Lie groups*, Japan J. Math. **39** (1970), 1–68.

b) *The Plancherel formula for* $SU(p, q)$, J. Math. Soc. Japan **22** (1970), 134–179. MR **42** #3230.

9. M. S. Narasimhan and K. Okamoto, *An analogue of the Borel-Weil-Bott theorem for hermitian symmetric pairs of noncompact type*, Ann. of Math. (2) **91** (1970), 486–511. MR **43** #419.

10. R. Parthasarathy,

a) *Dirac operator and the discrete series*, Ann. Math. **96** (1972), 1–30.

b) *A note on the vanishing of certain 'L^2-cohomologies'*, J. Math. Soc. Japan **23** (1971), 676–691.

11. W. Schmid, *On a conjecture of Langlands*, Ann. of Math. (2) **93** (1971), 1–42. MR **44** #4149.

12. L. Schwartz, *Théorie des distributions*. Tome II, Actualités Sci. Indust. no. 1122, Hermann, Paris, 1951. MR **12**, 833.

13. R. Steinberg, *Differential equations invariant under finite reflexion groups*, Trans. Amer. Math. Soc. **112** (1964), 392–400. MR **29** #4807.

14. P. C. Trombi and V. S. Varadarajan

a) *Spherical transforms on semisimple Lie groups*, Ann. of Math. (2) **94** (1971), 246–303. MR **44** #6913.

b) *Asymptotic behaviour of eigenfunctions on a semisimple Lie group: the discrete spectrum*, Acta Math. **129** (1972), 237–280.

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