

Criteria for the unitarizability of some highest weight modules*

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MS received 20 April 1979

Abstract. For a linear semisimple Lie group we obtain a necessary and sufficient condition for a highest weight module with non-singular infinitesimal character to be unitarizable.

Keywords. Highest weight module; spin module; formal Dirac operator; parabolic subalgebra; noncompact roots; infinitesimal character.

1. Introduction

A linear semisimple Lie group G admits nontrivial unitarizable highest weight modules precisely when it admits holomorphic discrete series. Supposing G is such a group, it is of interest to characterise the unitarizable ones among the set of all highest weight modules of G . We are looking for a condition which is both necessary and sufficient for a highest weight module π of G to be unitarizable. The most desirable (and one that would be the simplest) is to give a condition directly on the highest weight μ of the module π . The main results of this paper (theorem A, § 3 and theorem B, § 5) give such an explicit necessary and sufficient condition on μ , provided the infinitesimal character of π is nonsingular. In § 6, we discuss the applications of our results to the (o, p) Betti numbers of compact quotients of bounded domains.

Let G be a connected linear semisimple Lie group and let G_c be the complexification of G . Assume that G_c is simply connected. Let $\mathfrak{g}_0$ be the Lie algebra of G and let $\mathfrak{g}$ be the Lie algebra of G_c . Let K be a maximal compact subgroup of G . Let $\mathfrak{k}_0$ be the Lie algebra of K and $\mathfrak{k}$ the complexification of $\mathfrak{k}_0$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the Cartan decomposition and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be its complexification. We will denote by θ the corresponding Cartan involution.

We now assume that the symmetric space G/K is a hermitian symmetric domain. As is well-known $\mathfrak{p}$ can be canonically identified with the space of (complex) tangent vectors at the identity coset eK in G/K . Let $\mathfrak{p}_+$ be the subspace of $\mathfrak{p}$ consisting of the holomorphic tangent vectors at eK and $\mathfrak{p}_-$ the space of antiholomorphic tangent vectors at eK . It is well known that both $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are K submodules of $\mathfrak{p}$. Let b be a Cartan subgroup of $\mathfrak{k}$ and $\mathfrak{r}_b$ a Borel subalgebra of $\mathfrak{k}$ containing b . Then one knows that b is a Cartan subalgebra of $\mathfrak{g}$ and that $\mathfrak{r}_b + \mathfrak{p}_+$ is a Borel subalgebra of $\mathfrak{g}$.

* Partially supported by a grant from NSF (MCS 76-05962).

Let r be this Borel subalgebra of g . Let Δ be the set of roots of g with respect to b and let Δ_k and Δ_n be the sets of compact and non-compact roots respectively, so that

$$k = b + \sum_{\alpha \in \Delta_k} g^\alpha \text{ and}$$

$$p = \sum_{\alpha \in \Delta_n} g^\alpha$$

here g^α denotes the one dimensional space spanned by a root vector corresponding to α . Let P be the set of positive roots defined by the Borel subalgebra r . Thus

$$(1.1) \quad r = b + \sum_{\alpha \in P} g^\alpha.$$

P_k and P_n will denote respectively the set of compact and non-compact roots in P . We denote by δ , δ_k and δ_n half the sum of the roots in P , P_k and P_n respectively.

Let $U(g)$ be the enveloping algebra of g and $U(k)$ be the enveloping algebra of k . Let π be an irreducible smooth representation of G in a space $\tilde{H}$. Let H be the space of K finite vectors in $\tilde{H}$ so that $U(g)$ has an irreducible representation π in H for which H is $U(k)$ finite.

(1.2) *Definition* : π is said to be a highest weight module for g (or for G) if there exists $\mu \in b^*$ and $v \in H$, $v \neq 0$ such that

$$(1) \text{ For every } T \in b, \pi(T)v = \mu(T)v,$$

$$(2) \pi(X_\alpha)v = 0 \text{ if } \alpha \in P_k \text{ or } \alpha \in -P_n.$$

When there is some confusion, we will specify π is a highest weight module with respect to $P_k \cup -P_n$.

(1.3) *Definition* : π is said to be unitarizable if there exists a positive definite inner product $(\cdot, \cdot)$ on H , such that for every X in g_0 ,

$$(\pi(X)v, w) + (v, \pi(X)w) = 0 \text{ for all } v, w \text{ in } H.$$

Problem : Describe the set of all highest weight modules for G which are unitarizable.

We will focus our attention on the set of all highest weight modules for G which have a nonsingular infinitesimal character (see § 3 for definition).

If π is a highest weight module for G , then upto equivalence π is uniquely determined by its highest weight μ (Definition (1.2)) and μ is uniquely determined by π . Also μ satisfies

$$(1.4) \quad 2(\mu, \alpha)/(a, a) \in \mathbb{Z} \text{ for every } \alpha \in P$$

$$\text{and} \quad 2(\mu, \alpha)/(a, a) \in \mathbb{Z}^+ \text{ for every } \alpha \in P_k.$$

Moreover, to every μ satisfying (1.4) there corresponds (upto equivalence) a unique highest weight module π_μ of G whose highest weight is μ . This module is obtained as follows : Let V_μ be the finite dimensional irreducible module for K with highest weight μ . Regard V_μ as a module for $k + p_-$ by making the action of p_- trivial. Then the highest weight module π_μ is simply the unique irreducible quotient of $U(g) \otimes_{U(k+p_-)} V_\mu$.

We denote by H_μ this irreducible quotient. One can show that the action π_μ on H_μ comes as the action on K finite vectors of a suitable irreducible representation of G .

We note that $P_k \cup -P_n$ is the set of positive roots with respect to another lexicographic ordering on Δ . We denote by $\bar{P}$ this system of positive roots. Recall the g Verma modules $V_{\sigma, \bar{P}, \eta}$ with highest weight η (an element of b) relative to $\bar{P}$. Let $Z(g)$ denote the centre of the enveloping algebra $U(g)$. As is well-known elements of $Z(g)$ act by scalar multiplication on $V_{\sigma, \bar{P}, \eta}$. Let $\chi_{\bar{P}, \eta}$ denote the corresponding homomorphism of $Z(g)$ into C . If ω denotes the Casimir element in $Z(g)$, then it is known that

$$\chi_{\bar{P}, \eta}(\omega) = (\eta + \delta_{\bar{P}}, \eta + \delta_{\bar{P}}) - (\delta_{\bar{P}}, \delta_{\bar{P}})$$

where $\delta_{\bar{P}}$ denotes half the sum of the roots in $\bar{P}$. Thus

$$(1.5) \quad \chi_{\bar{P}, \eta}(\omega) = (\eta - \delta_n + \delta_k, \eta - \delta_n + \delta_k) - (\delta, \delta)$$

as $(\delta, \delta) = (\bar{\delta}, \bar{\delta})$

(1.6) *Corollary* : The Casimir acts on the highest weight module H_μ by the scalar

$$(\mu - \delta_n + \delta_k, \mu - \delta_n + \delta_k) - (\delta, \delta)$$

Proof : It is not hard to see that H_μ is precisely the irreducible quotient of $V_{\sigma, \bar{P}, \mu}$. Hence the result (q.e.d.).

2. An inequality satisfied for unitarizable representations

Let $\pi = \pi_\mu$ be an irreducible highest weight module for G . Let $H = H_\mu$ be the space of K finite vectors. We assume henceforth that π is unitarizable. For the results to be stated in this section, π can be an arbitrary irreducible unitary representation of G . Let L, L^+ and L^- be the spin module and the two half spin modules for $so(p)$ the Lie algebra of the special orthogonal group $SO(p)$ (The symmetric bilinear form on $p \subset g$ is the restriction of the Killing form). By composing with the adjoint action of k on p , we obtain the spin representation σ of k on L and the two half-spin representations $\sigma^\pm$ of k on L^+ and L^- . Recall that for every $x \in p$ there is a clifford multiplication $c(x) : L \rightarrow L$. Now, $H \otimes L$ is a k module and we have a formal Dirac operator $D : H \otimes L \rightarrow H \otimes L$ defined by

$$(2.1) \quad D = \sum \pi(X_i) \otimes c(X_i).$$

Here the summation is over an orthonormal basis for p_0 . There is a unique (upto a positive scalar multiple) positive definite inner product $(,)$ on L such that for every X in p_0 and s, s' in L ,

$$(2.2) \quad (c(x)s, s') + (s, c(x)s') = 0.$$

Since we have a positive definite inner product on H , for which also

$$(2.3) \quad (\pi(x)u, v) + (u, \pi(x)v) = 0$$

for every x in g_0 and for u, v in H , we now have a positive definite inner product on $H \otimes L$, the product of the ones on H and L . For u, v in H and s, s' in L , then

$$(u \otimes s, v \otimes s') = (u, v) (s, s').$$

With respect to this inner product we clearly have

$$(2.4) \quad (Dw, w') = (w, Dw')$$

for w, w' in $H \otimes L$.

Let ω_k be the Casimir element in $U(k)$. It equals $-\sum Y_i^2$ where Y_i is a basis of k_0 such that $(Y_i, Y_j) = \delta_{ij}$ where $(\cdot, \cdot)$ denotes the Killing form of g_0 .

A formula was obtained in [3, § 3] for the square of the Dirac operator. These computations also apply to the square of the formal Dirac operator (of also [7, § 8]). One thus obtains

$$(2.5) \text{ Lemma : } D^2 = (\pi \otimes \sigma)(\omega_k) - \pi(\omega) \otimes 1 - (\delta, \delta) + (\delta_k, \delta_k).$$

(2.6) *Proposition*: Assume that ξ is the highest weight of an irreducible k submodule of $H \otimes L$.

Then

$$(\xi + \delta_k, \xi + \delta_k) \geq (\mu - \delta_n + \delta_k, \mu - \delta_n + \delta_k).$$

Proof: Let w be an element of $H \otimes L$ contained in an irreducible k submodule of $H \otimes L$ with highest weight ξ . The Casimir ω_k of k acts on w by the scalar $(\xi + \delta_k, \xi + \delta_k) - (\delta_k, \delta_k)$. Thus by (2.5) and (1.6)

$$\begin{aligned} D^2 w &= (\xi + \delta_k, \xi + \delta_k) - (\delta_k, \delta_k) - (\mu - \delta_n + \delta_k, \mu - \delta_n + \delta_k) \\ &\quad + (\delta, \delta) - (\delta, \delta) + (\delta_k, \delta_k) \\ &= (\xi + \delta_k, \xi + \delta_k) - (\mu - \delta_n + \delta_k, \mu - \delta_n + \delta_k). \end{aligned}$$

Hence

$$(2.7) \quad (D^2 w, w) = \{(\xi + \delta_k, \xi + \delta_k) - (\mu - \delta_n + \delta_k)\} (w, w).$$

But $(D^2 w, w) = (DDw, w) = (Dw, Dw)$.

The last quantity is non-negative since the hermitian form on $H \otimes L$ is positive definite. For the same reason (w, w) is also nonnegative. Hence from (2.7) the assertion in the proposition follows.

(2.8) *Corollary*: Let π_μ be an irreducible highest weight module for G . Assume π_μ is unitarizable. Let V_μ be the irreducible finite dimensional module of k with highest weight μ . Suppose ξ is the highest weight of an irreducible k submodule of $V_\mu \otimes L$. Then,

$$(\xi + \delta_k, \xi + \delta_k) \geq (\mu - \delta_n + \delta_k, \mu - \delta_n + \delta_k).$$

Proof: This is clear from (2.6) since $V_\mu \subseteq H_\mu$.

3. A condition on μ

Let (π_μ, H_μ) be a highest weight module for G . In § 1, we observed that the centre $Z(g)$ of $U(g)$ acts on the Verma module $V_{\sigma, \bar{\sigma}, \eta}$ by the homomorphism $\chi_{\bar{\sigma}, \eta} : Z(g) \rightarrow C$. Any homomorphism χ of $Z(g)$ into C is of the form $\chi_{\bar{\sigma}, \eta}$ for a suitable element η in b^* . The homomorphism χ is said to be nonsingular if $\eta + \delta_{\bar{\sigma}}$ is nonsingular, i.e. $(\eta + \delta_{\bar{\sigma}}, a) \neq 0$ for any root a .

We will now assume that the infinitesimal character of π_μ is nonsingular. Since the infinitesimal character of π_μ is given by the homomorphism $\chi_{\bar{P}, \mu}$, our assumption amounts to making the hypothesis $\mu - \delta_n + \delta_k$ is nonsingular, i.e.

$$(3.1) \quad (\mu - \delta_n + \delta_k, a) \neq 0 \text{ for any root } a.$$

In addition, recall that the highest weights μ of highest weight modules for G satisfy the condition

$$2(\mu, a)/(a, a) \in \mathbb{Z} \text{ for every } a \text{ in } P \text{ and}$$

$$2(\mu, a)/(a, a) \in \mathbb{Z}^+ \text{ for every } a \text{ in } P_k.$$

Let P' be the set of roots defined by

$$(3.2) \quad P' = [a \in \Delta \mid (\mu - \delta_n + \delta_k, a) > 0].$$

Note that P' is the set of positive roots with respect to a lexicographic ordering. Also, observe that

$$(3.3) \quad 2(\mu - \delta_n + \delta_k, a)/(a, a) \text{ is a positive integer for every } a \text{ in } P'.$$

Let $\delta' =$ half the sum of the roots in P' . Then, note that

$$(3.4) \quad \mu - \delta_n + \delta_k = \lambda + \delta'.$$

Where λ satisfies

$$(3.5) \quad 2(\lambda, a)/(a, a) \text{ is a non-negative integer for every } a \text{ in } P'.$$

For every a in P_k , $(\mu, a) \geq 0$, $(-\delta_n, a) = 0$ and $(\delta_k, a) > 0$.

Hence

$$(3.6) \quad P' \supseteq P_k.$$

Let P'_n be the set of non-compact roots in P' and let δ' and let δ'_n be half the sum of the roots in P'_n .

Then $\delta' = \delta_k + \delta'_n$ and so (3.4) implies

$$(3.7) \quad \mu = \lambda + \delta_n + \delta'_n$$

Using our assumption that π_μ is unitarizable, we wish to conclude that the quantities λ and P'_n appearing above have very special properties. We now introduce some more terminology to explain this.

(3.8) Recall that r was the Borel subalgebra of g corresponding to the positive system P . Let q be a parabolic subalgebra of g containing r . Let

$$q = m + u$$

be the Levi decomposition of q such that m contains b . Thus u is the unipotent radical of q and m is a reductive component of q . Let P_m be the roots of (m, b) which are contained in P . Let P_n be the roots of P whose corresponding root-

spaces are contained in u . Thus P is the disjoint union of P_m and P_u . If P^- denotes the set $(-P_m) \cup P_u$ then it is known that P^- is also a positive system. Set P_n^- to be the set of non-compact roots in P^- .

(3.9) From the assumption that π_μ is unitarizable, we wish to conclude the following property about the expression $\mu = \lambda + \delta_n + \delta'_n$. *There exists a parabolic subalgebra q containing r such that with the notation introduced above $P'_n = P_n^-$ and $(\lambda, a) = 0$ for every a in P_m .*

(3.10) *Example* : Observe that if $(\mu - \delta_n + \delta_k, a) > 0$ for every a in P , then $P = P'$ and the above property is easily seen to hold by taking $q = r$ the Borel subalgebra itself. ($P_m = \text{empty}$ in this case and $P_n^- = P_n$). This is precisely the case when π_μ is a member of the holomorphic discrete series. As another illustration, we mention the case $\mu = 0$, so that π_μ is the trivial one dimensional representation, which is unitarizable. In this case, the property is seen to hold by taking $q = g$ and $\lambda = 0$ ($P_m = P$ in this case and $P_n^- = -P_n$).

(3.11) We will quickly see that $(\lambda, a) = 0$ for every a in $P \cap -P'$. Corollary (2.8) says that if ξ is the highest weight of an irreducible k submodule of $V_\mu \otimes L$, then $(\xi + \delta_k, \xi + \delta_k) \geq (\mu - \delta_n + \delta_k, \mu - \delta_n + \delta_k)$. The spin module L is self dual and one has knowledge about the highest weights of irreducible k submodules of L (cf. [3, §2]). Using these one can see that the irreducible k module with lowest weight $-\delta'_n$ occurs in L . Let us denote by $V_{-\delta'_n}$ this component contained in L . Then $V_\mu \otimes V_{-\delta'_n} \subseteq V_\mu \otimes L$. If we now take $\xi = \mu - \delta_n$, then since $\mu = \lambda + \delta_n + \delta'_n$ (cf. (3.7)), $\mu - \delta'_n \subseteq \lambda + \delta_n$. Both λ and δ_n are dominant and integral with respect to P_k . Thus there is an irreducible finite dimensional k module $V_{\mu - \delta'_n}$ with highest weight $\mu - \delta'_n$. By [6, 2.26] $V_{\mu - \delta'_n}$ occurs in $V_\mu \otimes V_{-\delta'_n}$. Applying corollary (2.8) to $\xi = \mu - \delta'_n = \lambda + \delta_n$ we conclude that

$$(\lambda + \delta_n + \delta_k, \lambda + \delta_n + \delta_k) \geq (\mu - \delta_n + \delta_k, \mu - \delta_n + \delta_k)$$

Since $\mu = \lambda + \delta_n + \delta'_n$, $\mu - \delta_n + \delta_k = \lambda + \delta'_n + \delta_k = \lambda + \delta'$ (cf. (3.4)).

Thus

$$(\lambda + \delta_n + \delta_k, \lambda + \delta_n + \delta_k) \geq (\lambda + \delta', \lambda + \delta')$$

$$\text{i.e. } (\lambda + \delta, \lambda + \delta) \geq (\lambda + \delta', \lambda + \delta')$$

$$\text{i.e. } (\lambda, \lambda) + 2(\lambda, \delta) + (\delta, \delta) \geq (\lambda, \lambda) + 2(\lambda, \delta') + (\delta', \delta').$$

But $(\delta, \delta) = (\delta', \delta')$. So we conclude that

$$(\lambda, \delta) \geq (\lambda, \delta')$$

That is $(\lambda, \delta' - \delta) \leq 0$.

But λ is dominant with respect to P' . Hence we conclude that $(\lambda, a) = 0$ for every a in $P \cap -P'$.

We wanted to show that there exists a parabolic subalgebra q containing r such that $P'_n = P_n^-$ and $(\lambda, a) = 0$ for every a in P_m . For any q if we define

$$P_{m,n} = \text{the set of non-compact roots in } P_m$$

then

$$P_{m,n} = P_n \cap -P_n^-.$$

Also, the assertion that $P_n^* = P_n^-$ is the same as the assertion

$$(3.12) \quad P_{m,n} = P_n \cap -P_n'.$$

We will now commence a long chain of arguments and eventually show that μ is of the very special type discussed in (3.9).

Suppose Y is a subset of $P_n \cap -P_n'$. We denote by q_Y the intersection of all parabolic subalgebras q of g containing r such that P_m contains Y .

(3.13) *Remark.* For each $Y \subseteq P_n \cap -P_n'$, q_Y has the following property. No semisimple ideal of the reductive part of q_Y is contained in k .

The reason is the following. As is well-known the parabolic subalgebras q of g containing r are in one to one correspondence with subsets of the set S of simple roots of P . Suppose q contains r and suppose $Y \subseteq P_m$, where P_m is the set of roots of P which belong to the reductive part m of q . Suppose m has a semisimple ideal m_1 such that $m_1 \subseteq k$. This is equivalent to the statement "Let $X \subseteq S$ be the subset of S corresponding to q . Then X can be written as a disjoint union $X_1 \cup X_2$ such that all the roots of X_1 are compact and X_1 is orthogonal to X_2 , i.e. $(\alpha, \beta) = 0$ for any α in X_1 and any β in X_2 ." But then if q_2 is the parabolic subalgebra of g containing r corresponding to $X_2 \subseteq S$, then q_2 is a proper subalgebra of q and the set of non-compact roots in the reductive part of q_2 is exactly the same as those in the reductive part of q . In particular, Y is still contained in the set of roots of the reductive part of q_2 , since Y contains only non-compact roots. Since q_Y is the intersection of all parabolic subalgebras q of g containing r for which $Y \subseteq P_m$, it is now clear that the reductive part of q_Y has no semisimple ideal contained in k . This completes the proof of (3.13).

Varying Y over the subsets of $P_n \cap -P_n'$ we get a collection of parabolic subalgebras

$$\{q_Y \mid Y \subseteq P_n \cap -P_n'\}.$$

It should be remarked that the set $P_{m,n}$ of those non-compact roots of P which are contained in the reductive part of any $q = q_Y$ may not be contained in $P_n \cap -P_n'$.

(3.14) Consider the collection of those parabolic subgroups $q = q_Y$, $Y \subseteq P_n \cap -P_n'$, for which $P_{m,n}$ is contained in $P_n \cap -P_n'$. (This set is non-empty since q_Y , when Y is the empty set is obviously a member). Among all such q_Y , choose one for which $P_{m,n}$ has maximum possible cardinality.

In what follows, unless otherwise stated, q will denote this particular parabolic subalgebra and the sets $P_{m,n}$, P_m , P_n^- , etc. (cf. (3.8)) shall all be with respect to this particular parabolic subalgebra q . We now set

$$(3.15) \quad P^1 = P_+ \cup P_n^-$$

(3.16) We claim that P^1 is a positive system in Δ .

The simplest way to prove this is through the following argument. Under the well-known one to one correspondence between parabolic subalgebras of g contain-

ing r and subsets of S , there is a unique subset X of S corresponding to our parabolic subalgebra q chosen. (The subset X is precisely the set of simple roots of P_m). Suppose for a while g is not only semisimple but actually simple. (This assumption is not really necessary but is made here only to illustrate the argument. The proof in the general case is alike). Then one knows that S contains only one simple non-compact root, say a_1 . Also it is known that the coefficient of a_1 in the highest root in P is one. This is *characteristic* of the hermitian symmetric case, where $P_k \cup P_n$ and also $P_k \cup -P_n$ are *both* positive systems. If a_1 belongs to X (which, as was observed before, is the set of simple roots for $P_m = P_{m,k} \cup P_{m,n}$) then a_1 is the only non-compact root in X and its coefficient in the highest root of P_m is one. Thus $P_{m,k} \cup (-P_{m,n})$ is a positive system for the set of roots of m with respect to b . If a_1 does not belong to X then $P_{m,n}$ is empty and so again $P_{m,k} \cup (-P_{m,n})$ is a positive system for the roots of m . If $\tilde{P}_m$ is any positive system for the roots of $\tilde{m}$, then $P_m \cup P_u$ (where P_u is the set of all roots in P , whose root-spaces are contained in the unipotent radical of q) is a positive system for the roots of b in g . But one sees easily that the set P^1 in (3.15) is nothing but

$$(3.17) \quad P^1 = (P_{m,k} \cup -P_{m,n}) \cup P_u \text{ (disjoint).}$$

Thus, the assertion (3.16) is proved. As we remarked, the case when g is not simple can be treated in the same way.

(3.18) We now let S^1 be the set of simple roots of P^1 . Let r^1 be the Borel subalgebra of g corresponding to P^1 . Since $P^1 = P_k \cup P_n^-$ (the latter defined with respect to q) one sees at once that r^1 is contained in q .

(3.19) Let X^1 be the subset of S^1 corresponding to q .

(3.20) We enumerate S^1 as $a_1, a_2, \dots, a_i, \dots, a_j$ in such a way that $X^1 = a_1, a_2, \dots, a_i$.

Remark : Even when g is simple S^1 may contain more than one non-compact root.

We now show that $P' = P^1$. This will be used in the proof of 3.9.

(3.21) Suppose P' is not equal to P^1 .

We wish to show that (3.21) leads to a contradiction, namely (3.34).

If P^1 is not equal to P' , then there is a simple root a in S^1 , such that $-a$ belongs to P' .

However P^1 and P' have some common parts. Let us look at this very carefully. First of all both P^1 and P' contain P_k (cf. (3.15) and (3.6)). Secondly, observe that by the choice of q made in (3.14), $P_n \cap P'_n \subseteq P_{u,n}$, where $P_{u,n}$ denotes the set of non-compact roots, whose root spaces are contained in the unipotent radical of q . But $P_{u,n} \subseteq P^1$ (cf. (3.17)). So, $P_n \cap P'_n \subseteq P_n^1$, where P_n^1 denotes the set of non-compact roots in P^1 . This means that all those roots which are common to $-P_n$ and $-P'_n$ are also common to $-P_n^1$ and $-P_n^{1'}$. In particular, a root common to $-P'_n$ and P_n^1 cannot be in $-P_n \cap -P'_n$; it has to lie in

$P_n \cap -P'_n$. The root a picked out in the beginning of this paragraph is not common to P^1 and P' . Thus, in view of the preceding observations, we can infer two facts about this root a . First, it cannot be a compact root; thus it has to lie in $P_n^1 \cap -P'_n$. So, secondly it cannot be in $-P_n \cap -P'_n$ but has to be in $P_n \cap -P'_n$. Without loss, we can assume that S^1 has been enumerated in (3.20), in such a way that $a = a_{i+1}$.

Thus,

(3.22) In the enumeration (3.20), a_{i+1} is a noncompact root and $a_{i+1} \in P_n \cap -P'_n$.

For any positive integer e such that $1 \leq e \leq j$, let q^e denote the parabolic subalgebra of g corresponding to the subset $\{a_1, a_2, \dots, a_e\}$ of S^1 . Thus $q^1 = q$ (cf. (3.20)) and q^{i+1} denotes the parabolic subalgebra of g corresponding to the subset $\{a_1, a_2, \dots, a_{i+1}\}$ of S^1 . Note that since q^{i+1} contains $q^1 = q$, *a fortiori*.

(3.23) q^{i+1} contains the Borel subalgebra r .

(3.24) We also claim that q^{i+1} is of the form q_Y described before (3.13) for a suitable subset $Y \subseteq P_n \cap -P'_n$.

In fact let Y^0 be the set $P_{m,n}$ of the non-compact roots in P which belong to the reductive part of our chosen q . Then $Y^0 \subseteq P_n \cap -P'_n$ and $q = q_{Y^0}$. If we now let $Y = Y^0 \cup \{a_{i+1}\}$ then in view of (3.22) $Y \subseteq P_n \cap -P'_n$ and it is easy to see that q^{i+1} equals the corresponding q_Y . Let $P_{m,n}^{i+1}$ denote the set of roots of the reductive part of q^{i+1} which belong to P . Let $P_{m,n}^{i+1}$ denote the set of non-compact roots in $P_{m,n}^{i+1}$. The reductive part of q^{i+1} contains the reductive part of $q^1 = q$ and is strictly bigger than the reductive part of q ; a_{i+1} is a noncompact root which belongs to the reductive part of q^{i+1} but it does not belong to the reductive part of q^1 . Thus, in fact, the set of non-compact roots in the reductive part of q^1 is a proper subset of the set of non-compact roots in the reductive part of q^{i+1} . But q was chosen to be maximal having a certain property stated in (3.14). Thus we conclude

(3.25) $P_{m,n}^{i+1}$ is not contained in $P_n \cap -P'_n$.

But, evidently, by very definition, $P_{m,n}^{i+1} \subseteq P_n$. Thus we conclude that

(3.26) there is a root β of $P_{m,n}^{i+1}$ which belongs to $P_n \cap P'_n$.

The root β will be the 'trump' in our 'reductio ad absurdum.' Since $P_n \cap P'_n \subseteq P^1$, (cf. arguments preceding (3.22)) β is a root in P^1 , hence a non-negative integral linear combination of the simple roots S^1 of P^1 . In particular,

(3.27) $\beta = A + da_{i+1}$, where A is a non-negative integral linear combination of the roots $a_1, \dots, a_i$ and d is a positive integer.

Note that many of the roots in $\{a_1, \dots, a_i\}$ may be compact. To proceed with the argument, we would like to show that A can actually be written as a nonnegative real linear combination of the set of non-compact roots of a positive system for Δ_m , the roots of the reductive part m of q . To this end we will prove a slightly more general result.

(3.28) *Lemma.* Suppose g_0 is any real semisimple Lie algebra. Let $g_0 = k_0 + p_0$ be a Cartan decomposition of g_0 . Assume that g_0 has no semisimple ideals contained in k_0 . Let b_0 be a Cartan subalgebra of k_0 and assume b_0 is also a Cartan subalgebra of g_0 . Let g, k, b , etc. be the complexifications. Let Δ be the set of roots of (g, b) . Let ϕ be any real linear form on $i b_0$. Then there exists a positive system P in Δ such that ϕ is a non-negative real linear combination of elements of P , the set of non-compact roots in P .

Proof. Start with any positive system P^0 in Δ . Let S^0 be the set of simple roots of P^0 . Let $S_0 = A_1 \cup A_2 \cup \dots \cup A_t$ be a partition of S^0 such that

A_t = all the non-compact roots in S^0 ,

A_{t-1} = all those compact roots in S^0 , which are connected (i.e. having a non-zero scalar product) with some element of A_t ,

A_{t-2} = all those compact roots in $S^0 - \{A_t \cup A_{t-1}\}$

which are connected to A_{t-1}

A_{t-3} = all those compact roots in $S^0 - A_t \cup A_{t-1} \cup A_{t-2}$

which are connected to A_{t-2} , etc.

Because of the assumption that g_0 has no compact factors the above procedure certainly exhausts all of S^0 .

Let $a_1, a_2, a_3, \dots$ be an enumeration of elements of $A_1, \beta_1, \beta_2, \beta_3, \dots$ an enumeration of elements of A_2 , etc.

Let $\phi = \sum_{\gamma \in S^0} m_\gamma \gamma$ be the unique expression for ϕ in terms of the basis elements $\{\gamma \mid \gamma \in S^0\}$; m_γ are real numbers, some negative and some non-negative. Without loss of generality we can assume that m_{a_1} , the coefficient of a_1 in ϕ is non-negative. Let q^1 be the parabolic subalgebra of g corresponding to the subset $S^0 - \{a_1\}$ of S^0 . Let g^1 be the reductive part of q^1 and u^1 , the unipotent radical of q^1 . Observe that there is at least one non-compact root of P^0 occurring in u^1 ; for, otherwise, $p \subseteq g^1 (\neq g)$ which can only happen if g_0 has compact semisimple ideals, contrary to what was assumed. Let ξ be a noncompact root of P^0 occurring in u^1 . In particular, ξ can be written as a non-negative integral linear combination of elements of S^0 , such that the coefficient of a_1 is positive. We now choose a non-negative real number c such that if

$$(3.29) \quad \phi - c\xi = \sum_{\gamma \in S^0} n_\gamma \gamma,$$

then $n_{a_1} = 0$.

(3.30) The complex Lie algebra g^1 is the complexification of the real Lie algebra $g_0^1 = g^1 \cap g_0$ and g_0^1 has no compact semisimple factors.

Let us postpone the proof of this but assume it for a while.

By (3.29), $\phi - c\xi$ is a real linear combination of roots of g^1 . Also, the semisimple rank of g^1 is strictly less than the semisimple rank of g . Thus, using a suitable induction hypothesis, we can assume that $\phi - c\xi$ is a non-negative real linear combination of the non-compact roots of some positive system $P^{(1)}$ of the roots of

g^1 with respect to b . This can be done because by (3.30), g_0^1 (of which g^1 is the complexification) satisfies the hypothesis of the lemma.

All we have to do now to prove the lemma is to enlarge $P^{(1)}$ to a positive system P of the roots of g with respect to b , by adjoining the roots which occur in the unipotent radical of q^1 . For then (3.29) achieves our aim.

It now remains to show (3.30) is true.

Suppose g_c is a compact form of a complex semisimple Lie algebra g and suppose q is a parabolic subalgebra of g . Then the intersection of q with g_c is a compact real form of a reductive part of q . This is well known. In our case $g_c = k_0 + iP_0$ is a compact real form of g . Let k_0^1 and p_0^1 be the intersection of g^1 with k_0 and p_0 respectively. We observe that q^1 and g^1 are both stable under the Cartan involution θ associated to the Cartan decomposition $g_0 = k_0 + p_0$. (The reason is this: since we assumed rank of $g_0 = \text{rank of } k_0$, θ is the inner automorphism of an element of $\exp b_0$; but $b_0 \subseteq m^1 \subseteq q^1$). In view of this remark it is not hard to see that the intersection of g^1 with g_0 is $k_0^1 + p_0^1$ and the intersection of g^1 with $g_c (= k_0 + iP_0)$ is $k_0^1 + ip_0^1$. Thus $g^1 \cap g_0$ is a real form of g^1 since $g^1 \cap g_c$ is so.

Now the real reductive Lie algebra $g_0^1 = k_0^1 + p_0^1$ has a Cartan subalgebra b_0 contained in k_0^1 and we can talk of compact and non-compact roots. The set $S^0 - \{a_1\}$ is the set of simple roots for an appropriate positive system for the roots of g^1 with respect to b . If g^1 has a semisimple ideal contained in k^1 , then $S^0 - \{a_1\}$ can be written as a disjoint union $X_1 \cup X_2$ such that all the roots of X_1 are compact and X_1 is orthogonal to X_2 . But this cannot be done as is seen by the way the a_1 was chosen.

This completes the proof of lemma (3.28).

Applying the result of (3.28) to the quantity A on the right hand side of the equality (3.27), we see that there is a positive system Q for the roots of m (the reductive part of the parabolic subalgebra q chosen in (3.14)) such that

$$A = \sum_{\gamma \in Q_n} m_\gamma \gamma$$

where γ runs through the set Q_n of non-compact roots in Q and m_γ are non-negative real numbers. Thus from (3.27) we obtain

$$(3.31) \quad \beta = da_{t+1} + \sum_{\gamma \in Q_n} m_\gamma \gamma.$$

Where d, m_γ are all non-negative real numbers.

By the choice of q (cf. (3.14)), for every non-compact root a in the reductive part of q , either a or $-a$ lies in $P_n \cap -P'_n$. Thus,

$$(3.32) \quad \text{If } Q_n = \{\gamma_1, \gamma_2, \dots, \gamma_t\} \text{ then either } \gamma_i \text{ or } -\gamma_i \text{ lies in } P_n \cap -P'_n.$$

We now enlarge Q to a positive system Q^* for the roots of g , by adjoining to Q the set P_n of roots in the unipotent radical of q . It should be remarked that Q^* may not contain P_k . Let δ_k^* (resp. δ_n^*) be half the sum of the compact roots (resp. non-compact roots) in Q^* . There is a unique element w of the Weyl group of k such

$$(3.33) \quad \delta_k = w^{-1} \delta_k^*.$$

Then $w^{-1} \delta_n^*$ is the highest weight of an irreducible component of the spin module L for k . Since L is selfdual, $w^{-1} (-\delta_n^*)$ is the lowest weight of an irreducible component of the spin module L . Clearly $-\delta_n^*$ is in the orbit under the Weyl group of k of this lowest weight. We denote this irreducible component by $V_{-\delta_n^*}$. The contradiction we are aiming at (from 3.21) is the following :

(3.34) There is an irreducible component V_ξ with highest weight ξ contained in $V_{\lambda+\delta_n+\delta'_n} \otimes V_{-\delta_n^*} \subseteq V_{\lambda+\delta_n+\delta'_n} \otimes L$ for which $(\xi + \delta_k, \xi + \delta_k)$ is strictly less than $(\lambda + \delta', \lambda + \delta')$ (compare with Corollary (2.8) and (3.4)).

The proof of (3.34) is also a little lengthy. We proceed as follows.

Let V_ϕ and V_τ be two irreducible finite dimensional modules for k with highest weights (with respect to P_k) ϕ and τ respectively. Let s_1 and s_2 be two elements of the Weyl group W_k of k . For any root a let s_a denote the element of W_k which corresponds to the reflection associated to a . Suppose there is an element $a \in P_k$ such that

$$s_2 = s_a s_1$$

and $N(s_2) = N(s_1) + 1$, where $N(s)$ denotes the length of s , i.e., the length of a minimal expression for s as the product of reflections associated to simple roots in P_k . For any $s \in W_k$, let $V_{\phi+s\tau}$ denote the unique irreducible finite dimensional representation whose highest weight lies in the orbit of $\phi + s\tau$. Let ω_k be the Casimir element in the enveloping algebra of k . Let C_s denote the constant by which ω_k acts on $V_{\phi+s\tau}$. We claim

$$(3.35) \quad C_{s_2} \leq C_{s_1}.$$

Also, let t be the unique element of the Weyl Group W_k such that $t(P_k) = -P_k$. Then we also claim

$$(3.36) \quad C_t \leq C_s \text{ for any } s \in W_k.$$

To show (3.35), it is enough to show that $\phi + s_2\tau$ is a weight of the irreducible module $V_{\phi+s_1\tau}$. The latter fact will follow from the following computation. On the one hand,

$$(3.37) \quad \begin{aligned} s_a(\phi + s_1\tau) &= s_a\phi + s_2\tau \\ &= \phi + s_2\tau - \frac{2(\phi, a)}{(a, a)}a. \end{aligned}$$

On the other hand, $s_a(\phi + s_1\tau) = \phi + s_1\tau - 2[(\phi, a)/(a, a)]a - 2[(s_1\tau, a)/(a, a)]a$. Therefore, using (3.37)

$$\phi + s_2\tau = \phi + s_1\tau - 2 \frac{(s_1\tau, a)}{(a, a)}a$$

Since τ is dominant with respect to P_k and since $N(s_a s_1) > N(s_1)$, $2(s_1\tau, a)/(a, a)$ is a nonnegative integer. Both $\phi + s_1\tau$ and $s_a(\phi + s_1\tau)$ are weights of $V_{\phi+s_1\tau}$. From what we said above and from (3.37) it follows that $\phi + s_2\tau$ is in between $s_a(\phi + s_1\tau)$ and $\phi + s_1\tau$ in the a -string of weights of $V_{\phi+s_1\tau}$ through $\phi + s_1\tau$. But the a -string of weights through a given weight of an irreducible module is

unbroken. Therefore $\phi + s_2\tau$ is a weight of $V_{\phi+s_2\tau}$. Thus, the claim (3.35) is proved.

Applying (3.35) successively (3.36) follows.

We will now apply (3.35) and (3.36) to $\phi = \lambda + \delta_n + \delta'_n$ and $\tau =$ the unique element in the orbit of $-\delta_n^*$ which is the highest weight of $V_{-\delta_n^*}$. In particular we can conclude the following.

(3.38) Let $s \in W_k$ be such that $s^{-1}(-\delta_n^*)$ is the highest weight of $V_{-\delta_n^*}$. Let $t \in W_k$ be such that $tP_k = -P_k$. Let C_s (resp. C_t) be the value of Casimir ω_k on $V_{\lambda+\delta_n+\delta'_n-\delta_n^*}$, the representation of k whose highest weight belongs to the orbit of $\lambda + \delta_n + \delta'_n - \delta_n^*$ (resp. value of ω_k on $V_{\lambda+\delta_n+\delta'_n+ts^{-1}(-\delta_n^*)}$).

Then

$$(3.39) \quad C_t \leq C_s.$$

The crucial observation in concluding the proof of (3.34), is the following lemma:

(3.40) *Lemma.* Let s be defined as in (3.38) and let C_s be the value of the Casimir ω_k on $V_{\lambda+\delta_n+\delta'_n-\delta_n^*}$. Then $C_s + (\delta_k, \delta_k)$ is strictly less than $(\mu - \delta_n + \delta_k, \mu - \delta_n + \delta_k)$.

(Recall $\mu = \lambda + \delta_n + \delta'_n$).

Proof: The root β chosen in (3.26) will play the key role in the proof. To understand the argument, we first consider the case $\lambda = 0$ and we investigate the value C_s of the Casimir ω_k on $V_{\delta_n+\delta'_n-\delta_n^*}$.

Let $\beta_1, \beta_2, \dots, \beta_f, \dots, \beta_r$ be the roots in P_n and let $-\beta_1, -\beta_2, \dots, -\beta_f, \beta_{f+1}, \dots, \beta_r$ be the roots in P'_n .

Recall the positive system Q for the roots of m , the reductive part of q , and the positive system Q^* for the roots of g , which was obtained by adjoining to Q the set P_n of roots in the unipotent radical of q . The set Q_n of noncompact roots in Q is described in (3.32). Also it is clear that every root in the unipotent radical of q belongs to P , since the Borel subalgebra of g defined by P is contained in q . From these descriptions, it is clear that the set Q_n^* , the set of non-compact roots in Q is contained in $\{\pm\beta_1, \pm\beta_2, \dots, \pm\beta_f, \beta_{f+1}, \dots, \beta_r\}$. The root a_{i+1} (cf. 3.22) is contained in $P_n \cap -P$. That is

$$a_{i+1} \in \{\beta_1, \beta_2, \dots, \beta_f\}.$$

Also, by our choice (cf. 3.20, 3.22) a_{i+1} is *not* a root in the set of non-compact roots in the reductive part m of q . Thus we can and do arrange the enumeration $\{\beta_1, \beta_2, \dots, \beta_f, \dots, \beta_r\}$ of P_n , so that in addition to the properties already mentioned, we also have,

$$(3.41) \quad Q_n = \{-\beta_1, -\beta_2, \dots, -\beta_f, \beta_{f+1}, \dots, \beta_r\}$$

where $f < j$

$$(3.42) \quad Q_n^* = Q_n \cup \{\beta_{f+1}, \dots, \beta_j, \beta_{j+1}, \dots, \beta_r\} \text{ and } \beta_j = a_{i+1} \text{ (cf. (3.22))}.$$

In addition we observe that the root β chosen in (3.26) belongs to $P_n \cap P'_n = \{\beta_{f+1}, \dots, \beta_r\}$. We can assume without loss that β is enumerated to be β_{f+1} .

Applying these notation and using (3.31) and (3.41) we obtain

$$(3.43) \quad \beta_{j+1} = -a_1\beta_1 - \dots - a_s\beta_s + a_{s+1}\beta_{s+1} + a_j\beta_j + a_l\beta_l$$

where a_i are non-negative real numbers.

With these preparations, we can now get back to analysing the value of the Casimir ω_k on $V_{\delta_n + \delta'_n - \delta''_n}$ the irreducible representation whose highest weight lies in the orbit of $\delta_n + \delta'_n - \delta''_n$. Note that

$$\delta_n = \frac{1}{2}(\beta_1 + \beta_2 + \dots + \beta_r)$$

$$\delta'_n = \frac{1}{2}(-\beta_1 - \dots - \beta_j + \beta_{j+1} + \dots + \beta_r)$$

$$\delta''_n = \frac{1}{2}(-\beta_1 \dots - \beta_s + \beta_{s+1} \dots + \beta_r).$$

Hence $\delta_n + \delta'_n - \delta''_n$ is given by

$$(3.44) \quad \delta_n + \delta'_n - \delta''_n = \frac{1}{2}(\beta_1 + \dots + \beta_s - \beta_{s+1} - \dots - \beta_j \\ - \dots - \beta_j + \beta_{j+1} \dots + \beta_r).$$

Several observations must be made from the expression on the right hand side of the equality in (3.44). First of all it shows that $\delta_n + \delta'_n - \delta''_n$ is a weight of the spin module L for k (cf. [3, § 2]). Moreover,

(3.45) $\frac{1}{2}(\beta_1 + \dots + \beta_s - \beta_{s+1} - \dots - \beta_j - \dots - \beta_j + \beta_{j+1} + \dots + \beta_r)$ is not in the orbits under W_k of the highest weights of irreducible components of L .

As we will see below, (3.45) will essentially follow from (3.43). A weight ϕ of the spin module L is in the orbit of the highest weight of some irreducible component of L if and only if

$$(3.46) \quad \phi = \frac{1}{2}(\gamma_1 + \dots + \gamma_r)$$

where $\{\gamma_1, \dots, \gamma_r\}$ is the set of noncompact roots in some positive system for the roots of g . It follows easily from (3.43) that whenever $\{\beta_1, \dots, \beta_s, -\beta_{s+1}, \dots, -\beta_j, -\beta_j\}$ is contained in a set $\{\gamma_1, \gamma_2, \dots, \gamma_r\}$ as described above then $-\beta_{j+1}$ also belongs to $\{\gamma_1, \dots, \gamma_r\}$. In particular, $\{\beta_1, \dots, \beta_s, -\beta_{s+1}, \dots, -\beta_j, \dots, -\beta_j, \beta_{j+1}, \dots, \beta_r\}$ cannot be the set of non-compact roots of a positive system for the roots of g . This is enough to conclude that $\frac{1}{2}(\beta_1 + \dots + \beta_s - \beta_{s+1} \dots - \beta_j \dots - \beta_j + \beta_{j+1} + \dots + \beta_r)$ is not in the W_k orbit of the highest weight of any irreducible component of L . One might wonder why can't $\frac{1}{2}(\beta_1 + \dots + \beta_s - \beta_{s+1} \dots - \beta_j \dots - \beta_j + \beta_{j+1} \dots + \beta_r)$ equal $\frac{1}{2}(\gamma_1 + \dots + \gamma_r)$ where $\gamma_1, \dots, \gamma_r$ is a set as described after (3.46). But if it were so, that would make the multiplicity of $\frac{1}{2}(\gamma_1 + \dots + \gamma_r)$ as a weight of L equal to at least two (cf. [3, § 2]) which by [3, § 2] again cannot happen.

Thus (3.45) is proved and hence by (3.44) $\delta_n + \delta'_n - \delta''_n$ is a weight of the spin module L for k , but $V_{\delta_n + \delta'_n - \delta''_n}$ is not an irreducible component of L .

We state now a general fact. Suppose ϕ is a weight of an irreducible finite dimensional module V_τ with highest weight τ . Assume that ϕ is not in the orbit of τ . Let V_ϕ be the irreducible module whose highest weight lies in the orbit of ϕ . Then the value of the Casimir ω_k on V_ϕ is strictly less than the value of ω_k on V_τ . To see this let $s\phi$ be the highest weight of V_ϕ where s is an element of W_k . Then $s\phi + \Sigma m_\alpha \alpha = \tau$ where $\Sigma m_\alpha \alpha$ is a nonnegative integral linear combination of the

roots in P_k , with at least one m_a different from zero. Thus $(\tau + \delta_k, \tau + \delta_k) = (s\phi + \delta_k, s\phi + \delta_k) + 2(s\phi + \delta_k, \Sigma m_a \alpha) + (\Sigma m_a \alpha, \Sigma m_a \alpha)$ which is strictly greater than $(s\phi + \delta_k, s\phi + \delta_k)$. Our assertion follows from this.

On every irreducible component of L , ω_k acts by $(\delta, \delta) - (\delta_k, \delta_k)$. Thus we conclude that,

(3.47) ω_k acts on $V_{\delta_n + \delta'_n - \delta_n^*}$ by a constant strictly less than $(\delta, \delta) - (\delta_k, \delta_k)$. This completes the proof of lemma (3.40) in the case $\lambda = 0$. In the general case we argue as follows.

Consider the irreducible finite dimensional module $V_{\lambda + \delta_n + \delta'_n - \delta_n^*}$ discussed preceeding (3.39).

$$\lambda + \delta_n + \delta'_n - \delta_n^* = \lambda + \frac{1}{2}(\beta_1 + \dots + \beta_s - \beta_{s+1} \dots - \beta_t \dots - \beta_j + \beta_{j+1} + \dots + \beta_r)$$

Let F_λ be the finite dimensional irreducible module for g whose highest weight lies in the orbit (under the Weyl group of g) of λ . Consider the k module $F_\lambda \otimes L$. Let $\tilde{P}$ be a positive system for the roots of g . Let $\tilde{\lambda}$ be the highest weight of F_λ with respect to $\tilde{P}$ and let $\tilde{\delta}_n$ be half the sum of the non-compact roots in $\tilde{P}$. Let $V_{\tilde{\lambda} + \tilde{\delta}_n}$ be the irreducible module for k whose highest weight lies in the orbit (under W_k) of $\tilde{\lambda} + \tilde{\delta}_n$. For each $\tilde{P}$, $V_{\tilde{\lambda} + \tilde{\delta}_n}$ occurs in $F_\lambda \otimes L$. On each one of the modules $V_{\tilde{\lambda} + \tilde{\delta}_n}$, the Casimir ω_k acts by the same constant, namely, $(\tilde{\lambda} + \tilde{\delta}, \tilde{\lambda} + \tilde{\delta}) - (\tilde{\delta}_k, \tilde{\delta}_k)$. For any other irreducible component V_ξ of $F_\lambda \otimes L$, the action of ω_k on V_ξ is strictly less than the above constant. No element in the orbit of $\lambda + \frac{1}{2}(\beta_1 + \dots + \beta_s - \beta_{s+1} \dots - \beta_t \dots - \beta_j + \beta_{j+1} + \dots + \beta_r)$ can be of the form $\tilde{\lambda} + \tilde{\delta}_n$ as described above, for the same reasons as we saw for the case $\lambda = 0$. Since $\lambda + \delta_n + \delta'_n - \delta_n^*$ is equal to $\lambda + \frac{1}{2}(\beta_1 + \dots + \beta_s - \beta_{s+1} \dots - \beta_t \dots - \beta_j + \beta_{j+1} + \dots + \beta_r)$ we conclude that ω_k acts on $V_{\lambda + \delta_n + \delta'_n - \delta_n^*}$ by a constant strictly less than $(\tilde{\lambda} + \tilde{\delta}, \tilde{\lambda} + \tilde{\delta}) - (\delta_k, \delta_k)$. The latter constant is simply $(\lambda + \delta', \lambda + \delta') - (\delta_k, \delta_k)$ since λ is the highest weight of F_λ with respect to P' . Thus the lemma (3.40) is completely proved.

Looking at (3.39) and the lines preceeding it and using lemma (3.40) we now conclude the following :

Let V_ξ be the irreducible component of $V_{\lambda + \delta_n + \delta'_n} \otimes V_{-\delta_n^*}$, whose highest weight ξ lies in the orbit of the sum of $\lambda + \delta_n + \delta'_n$ and $ts^{-1}(-\delta_n^*)$ which are respectively the highest weight of $V_{\lambda + \delta_n + \delta'_n}$ and the lowest weight of $V_{-\delta_n^*}$ (cf. [6]). Then the Casimir ω_k acts on V_ξ by a constant which is strictly less than $(\lambda + \delta', \lambda + \delta') - (\delta_k, \delta_k)$. In other words $(\xi + \delta_k, \xi + \delta_k)$ is strictly less than $(\lambda + \delta', \lambda + \delta')$.

This completes the proof of (3.34).

In view of corollary (2.8) the statement (3.34) clearly implies a contradiction. Thus, the assumption (3.21), namely, that P' is not equal to P^1 (c.f. (3.15) for the definition of P^1) leads to (3.34) which in turn leads to a contradiction. Thus, we have proved $P' = P^1$. In particular, $P'_n = P_n^1$, which by (3.15) is equal to P_n^- , the latter being defined with respect to q (cf. (3.8)).

We have thus obtained a very explicit necessary condition on the parameter μ of an irreducible highest weight module π_μ of G , in order for π_μ to be unitarizable. We have obtained this under the assumption that π_μ has a nonsingular infinitesimal character. We gather below the basic notation introduced in the course of our proof.

We denote by r the Borel subalgebra of g defined by the positive system P . For a parabolic subalgebra q of g containing r , we denote by m the unique reductive part of q containing the Cartan subalgebra b and call it the reductive part of q . Let P_m be the set of roots in P which are roots of the reductive part of q .

Theorem A. Let π_μ be an irreducible highest weight module for G which has highest weight μ (with respect to $P_k \cup -P_n$). Suppose that the infinitesimal character of π_μ is nonsingular. Now assume that π_μ is unitarizable. Then there exists a parabolic subalgebra q of g containing r such that

$$\mu = \lambda + 2\delta_{a,n}$$

where (i) $\delta_{a,n}$ is half the sum of the non-compact roots in the unipotent radical of q , (ii) $2(\lambda, a)/(\alpha, \alpha)$ is a non-negative integer for all a in P and (iii) $(\lambda, a) = 0$ for every root a in the reductive part of q .

Proof: Let P' be the positive system on which $\mu - \delta_n + \delta_k$ is positive. Then

$$(3.48) \quad \mu = \lambda + \delta_n + \delta'_n$$

where λ is an integral linear form, dominant with respect to P' and where δ'_n is half the sum of the non-compact roots in P' (cf. (3.7)). Let P'_n be the set of non-compact roots in P' . For each $Y \subseteq P_n \cap -P'_n$, let q_Y be the intersection of all parabolic subalgebras q of g containing r , such that Y is contained in the set of roots in the reductive part of q . For certain subsets Y , the set $P_n \cap P'_n$ is contained in the set of roots in the unipotent radical of q_Y . Choose a maximal one with this property and call this parabolic subalgebra q .

(3.49) For this q , we claim $P_n \cap P'_n$ is precisely the set of noncompact roots in the unipotent radical of q .

If this were not the case, we obtained a contradiction to the property (2.8) of unitarizable representations. Namely, we obtained (3.34). Thus, the assertion (3.49) is proved. It is therefore clear that $\delta_n + \delta'_n = 2\delta_{a,n}$. Hence by (3.48) $\mu = \lambda + 2\delta_{a,n}$. It remains to show the property, (ii) and (iii) for λ .

In (3.11), we proved that $(\lambda, a) = 0$ for every a in $P_n \cap -P'_n$. Because of (3.49) $P_n \cap -P'_n$ is precisely the set of noncompact roots in P_m . We also know that the reductive part of q has no semisimple ideals contained in k (cf. (3.13)). Thus every compact root of m is a linear combination of noncompact roots in P_m . Thus $(\lambda, a) = 0$ for every root a in P_m . This proves (iii). Note that

$$P = P_n \cup (P_n \cap P'_n) \cup (P_n \cap -P'_n).$$

Since λ is dominant with respect to P' and since P_k as well as $P_n \cap P'_n$ are contained in P' , $(\lambda, a) \geq 0$ if $a \in P_k \cup (P_n \cap P'_n)$. If $a \in P_n \cap -P'_n$, then we already saw (cf. (3.11)) that $(\lambda, a) = 0$. Thus (ii) is proved. This completes the proof of theorem A.

In the next two sections, we will see that the converse of Theorem A is also true.

4. The sufficiency of the condition

The purpose of this section and the next one is to prove the following theorem, which is converse to theorem A.

Theorem B. Let q be a parabolic subalgebra of g containing r . Let $\delta_{q,n}$ be half the sum of the non-compact roots in the unipotent radical of q . Let λ be a linear form such that $2(\lambda, a)_{(a,a)}$ is a nonnegative integer for every a in P and such that $(\lambda, a) = 0$ for every root a in the reductive part of q . Let $\mu = \lambda + 2\delta_{q,n}$. Then the highest weight module (π_μ, H_μ) is unitarizable.

(4.1) *Remark.* π_μ , as in Theorem B, will have a nonsingular infinitesimal character. The first part in the proof of Theorem B is the following.

(4.2) *Proposition.* Let (π_μ, H_μ) be as in Theorem B. Let L be the spin module for k . Let ξ be the highest weight of an irreducible k submodule of $H \otimes L$. Then $(\xi + \delta_k, \xi + \delta_k) \geq (\mu - \delta_n + \delta_k, \mu - \delta_n + \delta_k)$. Moreover, if V_ϕ is an irreducible k submodule of H_μ with highest weight $\phi \neq \mu$ and if V_ξ is an irreducible k submodule of $V_\phi \otimes L$ with highest weight ξ , then we actually have strict inequality $(\xi + \delta_k, \xi + \delta_k) > (\mu - \delta_n + \delta_k, \mu - \delta_n + \delta_k)$.

Proof: Our idea in proving (4.2) is to use the construction [5, §4] where one builds a chain of $U(g)$ modules above g -Verma modules and takes a quotient of the biggest object of the chain to obtain modules like H_μ . We explain this a little more now.

For the discussion below, the condition that the positive system P is adapted to the complex structure on G/K is not needed. In fact G/K need not even admit any invariant complex structure and P could be arbitrary.

For the parabolic subalgebra q of g , let P_u be the set of roots in the unipotent radical of q and let P_m be the set of elements in P which are roots of the reductive part of q . Thus $P = P_m \cup P_u$ (disjoint). Also it is known that $(-P_m) \cup P_u$ is also a positive system for the roots of g . Let σ be the unique element of the Weyl group of g such that $\sigma P = (-P_m) \cup P_u$. For a while, let η be any regular integral linear form dominant with respect to $(-P_m) \cup P_u$. Set $W_1 = V_{\sigma, P, -\eta - \delta}$. Let X equal the set of all simple roots of P which are elements of P_m . (Thus, X is simply the set of all simple roots of P_m). For each $a \in X$, $2(-\eta - \delta, a)_{(a,a)}$ is a non-negative integer. Hence for each $a \in X$, the Verma module $V_{\sigma, P, -\eta - \delta - a}$ is a proper submodule of $V_{\sigma, P, -\eta - \delta}$. In fact, if we set W_0 equal to the sum $\sum_{a \in X} V_{\sigma, P, -\eta - \delta - a}$, then W_0 is a proper submodule of $W_1 = V_{\sigma, P, -\eta - \delta}$. In the construction of [5] one builds a (finite) canonical chain of $\cup (g)$ modules containing W_1 . The maximal object of this chain has a unique irreducible quotient. Let us here call it D_η . In [5] it is shown that D_η is a k -finite $\cup (g)$ module. From the work of [1] and [5] the module D_η , among other properties, has the following properties relating it to $V_{\sigma, P, -\eta - \delta}$.

Let $P_{m,k}$ (resp $P_{u,k}$) be the compact roots in P_m (resp P_u). Then $P_k = P_{m,k} \cup P_{u,k}$. One knows that $(-P_{m,k}) \cup P_{u,k}$ is also a positive system for the roots of k .

Let τ (resp. t) be the unique element of the Weyl group W_k of k (regarded as a subgroup of the Weyl group of g) such that $\tau P_k = (-P_{m,k}) \cup P_{u,k}$ (resp. $tP_k = -P_k$). For $s \in W_k$, set $s'\phi = s(\phi + \delta_k) - \delta_k$. In [5] it is shown that

(4.3) $(t\tau)'(-\eta - \delta)$ is the highest weight of a k submodule of D_η , with multiplicity one.

(4.4) If ϕ is the highest weight of any k submodule of D_η , then ϕ is of the form $\phi = (t\tau)'(-\eta - \delta - A)$, where A is a nonnegative integral linear combination of elements of P and in addition $-\eta - \delta - A$ is a P_k extreme weight (cf. [1, § 2]) of W_1/W_0 .

For (4.3), see [5, § 5] and for (4.4) see [5, Prop. 4.4].

(4.5) Now suppose that $(-\eta - \delta, a) = 0$ for every a in X .

Let u^- be the span of the root spaces not contained in q . Then

$$(4.6) \quad g = u^- \oplus m \oplus u$$

where m (resp. u) is the reductive part of q (resp. the unipotent radical of q). If $C_{-\eta-\delta}$ is the one-dimensional weight space of W_1/w_0 with weight $-\eta - \delta$, then the condition that $(-\eta - \delta, a) = 0$ for every a in X ensures that $u \cdot C_{-\eta-\delta} = 0$ and $m \cdot C_{-\eta-\delta} \subseteq C_{-\eta-\delta}$. Let $U(u^-)$, $U(m)$ and $U(u)$ denote respectively the enveloping algebras of u^- , m and u . Then $U(m) \cdot U(u) \cdot C_{-\eta-\delta} = C_{-\eta-\delta}$. Thus (4.6) implies $U(u^-) \cdot C_{-\eta-\delta} = W_1/w_0$. In particular, any weight of W_1/w_0 is of the form $-\eta - \delta - A$, where A is a non-negative integral linear combination of elements of P_u , the roots which occur in the unipotent radical of q .

Applying these remarks to (4.4) we obtain the following.

(4.7) Suppose $(-\eta - \delta, a) = 0$ for every a in X . If ϕ is the highest weight of any k submodule of D_η , then ϕ is of the form $(t\tau)'(-\eta - \delta - A)$ where A is a nonnegative integral linear combination of elements in P_u .

Now let V_ϕ be an irreducible k submodule of D_η with highest weight ϕ and let $\phi = (t\tau)'(-\eta - \delta - A)$ as in (4.7). Let V_ξ be an irreducible k submodule of $V_\phi \otimes L$ with highest weight ξ .

We wish to conclude that

$$(4.8) \quad (\xi + \delta_k, \xi + \delta_k) \geq (\eta, \eta).$$

Let ψ be the lowest weight of an irreducible component of L . It is enough to prove the inequality (4.8) when ξ is in the W_k orbit of $\phi + \psi$. We know ψ is of the form $t\tilde{\delta}_n$, where $\tilde{\delta}_n$ is half the sum of the non-compact positive roots of some positive system $\tilde{P}$ for g such that $P_k \subseteq \tilde{P}$. Also, since ξ is dominant with respect to P_k and lies in the orbit of $\psi + t\tilde{\delta}_n$, it can be shown that for any $w \in W_k$,

$$(\xi + \delta_k, \xi + \delta_k) \geq (w(\phi + t\tilde{\delta}_n) + \delta_k, w(\phi + t\tilde{\delta}_n) + \delta_k).$$

Thus to prove (4.8) it suffices to show that for some $w \in W_k$,

$$(4.9) \quad (w(\phi + t\tilde{\delta}_n) + \delta_k, w(\phi + t\tilde{\delta}_n) + \delta_k) \geq (\eta, \eta).$$

Now

$$\begin{aligned}\phi &= (t\tau)'(-\eta - \delta - A) = t\tau(-\eta - \delta - A + \delta_k) - \delta_k \\ &= t\tau(-\eta - \delta - A) + t(\tau\delta_k + \delta_k).\end{aligned}$$

We will show (4.9) for the element $w = t\tau t$.

$$t\tau t\phi = t(-\eta - \delta - A) + t(\delta_k + \tau\delta_k).$$

Thus,

$$t\tau t(\phi + t\tilde{\delta}_n) = t(-\eta - \delta - A) + t(\delta_k + \tau\delta_k) + t\tau\tilde{\delta}_n.$$

So,

$$(4.10) \quad t\tau t(\phi + t\tilde{\delta}_n) + \delta_k = t(-\eta - \delta - A) + t\tau\delta_k + t\tau\tilde{\delta}_n.$$

In view of (4.10) to show (4.9) it is enough to show

$$(4.11) \quad (-\eta - \delta - A + \tau\delta_k + \tau\tilde{\delta}_n, -\eta - \delta - A + \tau\delta_k + \tau\tilde{\delta}_n) \geq (\eta, \eta).$$

Recall that $\tilde{\delta}_n$ was half the sum of the non-compact roots of a positive system $\tilde{P}$ such that $P_k \subseteq \tilde{P}$. Let Δ_m be the set of all roots of m . Then $\tilde{P} \cap \Delta_m$ gives a positive system $\tilde{P}_m$ for Δ_m and clearly $P_{m,k} \subseteq \tilde{P}_m$. Let $\tilde{P}^*$ = the positive system for the roots of g obtained by adjoining to $\tilde{P}_m$ the set P_u . Set $\tilde{\delta}_n^*$ = half the sum of the non-compact roots in $\tilde{P}^*$. Clearly,

$$(4.12) \quad \tilde{\delta}_n = \tilde{\delta}_n^* - B, \text{ where } B \text{ is a sum of elements from } P_{u,n}, \text{ the non-compact roots in } P_u.$$

Also note that

$$(4.13) \quad \tilde{P}_k \subseteq \tilde{P}^* \text{ and } P_u \subseteq \tilde{P}^*.$$

Every element of the Weyl group of m leaves P_u stable. In particular $\tau P_u \subseteq P_u$. In view of these remarks, the two positive systems P and $\tau\tilde{P}^*$ both contain P_u and differ only in the roots of m . In particular if s is the unique element of the Weyl group of g such that $sP = \tau\tilde{P}^*$, then s is actually an element of the Weyl group of m . Hence

$$(4.14) \quad sP_u = P_u \text{ and } s \text{ can be written as a product of reflections } s_a \text{ where the roots } a \text{ are in } X.$$

Note also that $s\delta = \tau\delta_k + \tau\tilde{\delta}_n^*$. We write it as

$$(4.15) \quad s\delta = \delta + (-\delta + \tau\delta_k + \tau\tilde{\delta}_n^*)$$

and think of it as the result obtained by applying the formula $s_a(\lambda) = \lambda - 2(\lambda, a)/(a, a)$ successively to the reflections s_a in the expression for s as in (4.14). Since $(-\eta, a) = (\delta, a)$ for every a in X , it follows from (4.15) that

$$(4.16) \quad s(-\eta) = -\eta - \delta_k + \tau\delta_k + \tau\tilde{\delta}_n^*.$$

With these preparations we can now show (4.11). Using (4.12) and (4.16), we see that

$$(4.17) \quad -\eta - \delta - A + \tau\delta_k + \tau\tilde{\delta}_n = s(-\eta) - A - \tau B$$

where A and τB are both nonnegative integral linear combinations of elements from P_n . We assumed that η was regular and dominant with respect to $(-P_m) \cup P_n$. Since $sP_n = P_n$, we see that $s(-\eta)$ is dominant with respect to $(sP_m) \cup (-P_n)$. Thus, from (4.17) we see that

$$(-\eta - \delta - A + \tau\delta_k + \tau\tilde{\delta}_n, -\eta - \delta - A + \tau\delta_k + \tau\tilde{\delta}_n) \geq (\eta, \eta)$$

and equality occurs only if $A = 0$ and $B = 0$.

Thus, we have shown (4.8); in fact we also proved that for equality to hold in (4.8), it is necessary that $\phi = (t\tau)'(-\eta - \delta)$, i.e., V_ϕ should be the unique 'minimal' k -type of D_η .

We now apply these general facts to our special case to prove proposition (4.2). First of all observe that for μ as in theorem B, the statements in proposition 4.2 for (π_μ, H_μ) will follow if we prove the corresponding statements for (π_μ^*, H_μ^*) the dual of (π_μ, H_μ) . We will prove the statements for (π_μ^*, H_μ^*) by identifying H_μ^* with a ' D_η ', described above. In fact, choose $\eta = \lambda - \delta_m + \delta_n$.

Then, we claim

$$(4.18) \quad D_\eta \approx H_\mu^*.$$

We will first verify

$$\begin{aligned} (4.19) \quad (t\tau)'(-\eta - \delta) &= -t\mu \\ (t\tau)'(-\eta - \delta) &= t\tau(-\eta - \delta + \delta_k) - \delta_k \\ &= t(\tau(-\eta - \delta + \delta_k) + \delta_k) \\ &= t(\tau(-\eta - \delta_m - \delta_n + \delta_k) + \delta_k) \\ &= t(\tau(-\lambda - 2\delta_n) + \tau\delta_k + \delta_k) \\ &= t(-\lambda - 2\delta_n + \tau\delta_k + \delta_k) \end{aligned}$$

since $\tau P_n = P_n$ and τ is a product of reflections s_a , $a \in X$, and $(\lambda, a) = 0$ for $a \in X$.

But $-2\delta_n + \tau\delta_k + \delta_k = -2\delta_{a,n}$.

So, $t(-\lambda - 2\delta_n + \tau\delta_k + \delta_k) = t(-\lambda - 2\delta_{a,n}) = -t\mu$ and this proves (4.19). Now, let ϕ be the highest weight of an irreducible k submodule of D_η . By (4.4) ϕ is of the form $\phi = (t\tau)'(-\eta - \delta - A)$ where A is nonnegative integral linear combination of elements of P . Here, in addition $-\eta - \delta - A$ should be a P_k extreme weight for W_1 . If V_1 denotes the k -Verma module $V_{k, P_k - \eta - \delta}$ contained in $W_1 = V_{s, P, -\eta - \delta}$, then the action of the enveloping algebra gives a k module surjection $U(\mathfrak{p}) \otimes V_1 \rightarrow W_1$. Hence any P_k extreme vector of W_1 has to be of the form $-\eta - \delta - A$ where A is a nonnegative integral linear combination of elements of P_n . Since elements of W_k leave P_n stable, we now conclude that ϕ is of the form $(t\tau)'(-\eta - \delta) - B$ where B is a nonnegative integral linear combination of elements of P_n . This shows that D_η^* is a highest weight module

(with respect to $P_k \cup -P_n$ as in Def. (1.1)) with highest weight $-t((t\tau)'(-\eta - \delta))$. Since $(t\tau)'(-\eta - \delta) = -t\mu$ (by (4.19)) the assertion (4.18) follows.

To conclude the inequalities and the statements in proposition (4.2), it remains to verify $(-\eta - \delta, a) = 0$ for a in X . But

$$-\eta - \delta = -\lambda + \delta_m - \delta_u - (\delta_m + \delta_u) = -\lambda - 2\delta_u.$$

But by assumption $(\lambda, a) = 0$, for a in X and also it is well known that $(2\delta_u, a) = 0$ for a in X .

This completes the proof of proposition (4.2). In the next section, we will use the result of proposition (4.2) and arrive at the unitarizability of (π_μ, H_μ) .

5. Role of the formal Dirac operator

The purpose of this section is to prove a general unitarizability result for highest weight modules when one knows an inequality as in proposition (4.2).

Let (π_μ, H_μ) be an irreducible highest weight module which admits an invariant hermitian form.

(5.1) *Proposition.* Let ϕ be the highest weight of an irreducible k submodule V_ϕ of H_μ and suppose that for every k submodule V_ξ contained in $V_\phi \otimes L$, one has $(\xi + \delta_k, \xi + \delta_k) \geq (\mu - \delta_n + \delta_k, \mu - \delta_n + \delta_k)$, with strict inequality whenever $\phi \neq \mu$. Then (π_μ, H_μ) is unitarizable.

The techniques of proving (5.1) are essentially the same as already employed in the proof of [4, Prop. 9.7]. For the benefit of the reader, we will discuss below the main ingredients of that argument.

We define a filtration H_i in $H = H_\mu$ as follows. H_0 is the irreducible k module V_μ . We inductively define $H_{i+1} = H_i + p_+ H_i$. Since H is a highest weight module, $H = U_i H_i$. Now, let us normalize the hermitian forms on H , so that it restricts to a positive definite one on H_0 . Inductively, assume that it restricts to a positive definite one on H_i . We wish to prove it restricts to a positive definite one on H_{i+1} .

The main tool we employ is the formal Dirac operator $D : H \otimes L \rightarrow H \otimes L$, (cf. (2.1)). Clearly $D(H_i \otimes L) \subseteq H_{i+1} \otimes L$. Let $L_{-\delta_n}$ be the one dimensional k submodule of L whose highest is $-\delta_n$.

$$(5.2) \text{ Lemma. } D(H_{i+1} \otimes L_{-\delta_n}) \subseteq H_i \otimes L.$$

With suitable normalizations,

$$(5.3) D = \sum_{a \in P_n} \pi(X_a) \otimes C(X_{-a}) + \sum_{a \in P_n} \pi(X_{-a}) \otimes C(X_a).$$

Clearly the part $\sum_{a \in P_n} \pi(X_a) \otimes C(X_{-a})$ annihilates $H_{i+1} \otimes L_{-\delta_n}$. Hence to prove (5.2) it is enough to show that $p_- \cdot H_{i+1} \subseteq H_i$. This can be done easily. Thus (5.2) is proved.

Take the standard hermitian form on L . Then we have a product hermitian form on $H \otimes L$. One can show that if $v, w \in I_\xi \subseteq H \otimes L$, where I_ξ is the isotypical k submodule of $H \otimes L$ with highest weight ξ , then,

$$(5.4) (Dv, Dw) = \{(\xi + \delta_k, \xi + \delta_k) - (\mu - \delta_n + \delta_k, \mu - \delta_n + \delta_k)\} (v, w).$$

Let H^+ be the unique k -submodule of H_i which is a complement of H_0 . Because of the hypothesis of the proposition (5.1) the scalar within brackets in (5.4) is a positive number whenever $v, w \in (U_i H_i^+) \otimes L$. It now follows from (5.4) and (5.2) and the induction hypothesis that the hermitian form on $H_{i+1}^+ \otimes L_{-\delta_n}$ is positive definite. Dividing out by the +ve factor coming from $L_{-\delta_n}$, we see that on H_{i+1}^+ the form is +ve definite. Since H_0 and H_{i+1}^+ are orthogonal, it now follows that on H_{i+1} the form is +ve definite.

This proves proposition (5.1).

Now, combining together proposition (4.2) and proposition (5.1), we have proved theorem B, since the modules in question are known to possess invariant hermitian forms.

6. Applications to (o, p) Betti numbers : Remarks

Let Γ be a discrete subgroup of G so that $\Gamma \backslash G / K$ is a compact locally symmetric hermitian domain. The (o, p) Betti number of $\Gamma \backslash G / K$ is a certain sum over the class of irreducible unitary highest weight modules (π, H) for G , having the same infinitesimal character as the trivial one-dimensional representation of G (cf. [2]). If such a module (π, H) has a nonzero contribution to the (o, p) Betti number, then necessarily,

$$(6.1) \quad \dim(\text{Hom}_k(\wedge^p p_+, H)) \neq 0.$$

In particular, (π, H) has to be a module for the adjoint group of g_0 . Moreover, since (π, H) has the same infinitesimal character as the trivial one-dimensional representation of G , the infinitesimal character of π is regular. Using theorem A and theorem B, we obtain the following.

(6.2) *Proposition.* Let q be a parabolic subalgebra of g containing r . Let $\mu = 2\delta_{a,n}$, the sum of all the non-compact roots in the unipotent radical of q . The highest weight modules (π_μ, H_μ) obtained this way for the various q consist precisely of the set of irreducible unitary highest weight modules for G having the same infinitesimal character as the trivial one-dimensional module.

(6.3) Let q be as in proposition (6.2) and let $\mu = 2\delta_{a,n}$. When is $\text{Hom}_k(\wedge^p p_+, H_\mu)$ nonzero? It is nonzero if and only if p is exactly the number of non-compact roots in the unipotent radical of q .

Suppose p is the number of non-compact roots in q and let $X_{a_1}, \dots, X_{a_p}$ be the corresponding root vectors. The vector $X_{a_1} \wedge \dots \wedge X_{a_p}$ in $\wedge^p p_+$ has weight μ . If β is a positive compact root, then for $1 \leq i \leq p$, either $[X_\beta, X_{a_i}] = 0$ or else, $[X_\beta, X_{a_i}]$ is a scalar multiple of X_{a_j} , $1 \leq j \leq p, j \neq i$. Hence $\text{ad}(X_\beta)(X_{a_1} \wedge \dots \wedge X_{a_p}) = 0$. Thus $X_{a_1} \wedge \dots \wedge X_{a_p}$ is a highest weight vector with highest weight μ . This proves $\text{Hom}_k(\wedge^p p_+, H_\mu) \neq 0$.

Conversely, suppose $\text{Hom}_k(\wedge^p p_+, H_\mu) \neq 0$. Then there exists an irreducible k module V_ϕ with highest weight ϕ such that $V_\phi \subseteq \wedge^p p_+$ and $V_\phi \subseteq H_\mu$. Since $\wedge^p p_+ \subseteq L \otimes L^*$, we have $\text{Hom}_k(V_\phi \otimes L, L) \neq 0$. Hence we can find an irreducible k module V_ξ with highest weight ξ such that $V_\xi \subseteq V_\phi \otimes L$ and $V_\xi \subseteq L$. Since $V_\xi \subseteq L$, $(\xi + \delta_k, \xi + \delta_k) = (\delta, \delta)$ (cf. [3, § 2]).

Note that since H_μ has the same infinitesimal character as the trivial one-dimensional module, $(\mu - \delta_n + \delta_k, \mu - \delta_n + \delta_k) = (\delta, \delta)$.

Thus $(\xi + \delta_k, \xi + \delta_k) = (\mu - \delta_n + \delta_k, \mu - \delta_n + \delta_k)$. As we already have $V_\xi \subseteq V_\phi \otimes L$ and $V_\phi \subseteq H_\mu$, we conclude from proposition (4.2) that $V_\phi = V_\mu$. Thus, $\text{Hom}_k(\wedge^p p_+, V_\mu) \neq 0$. But if s is the number of non-compact roots of q , then as we already saw $\text{Hom}_k(\wedge^s p_+, V_\mu) \neq 0$. Thus, $\text{Hom}_k(\wedge^p p_+, \wedge^s p_+) \neq 0$. This can happen only if $p = s$. Thus (6.3) is proved.

Since the multiplicity of V_ϕ in $\wedge^p p_+$ can be at most one, in the course of the above argument, we have actually proved

(6.4) The space $\text{Hom}_k(\wedge^p p_+, H_\mu)$ in (6.3) has dimension exactly one.

The Betti numbers of $\Gamma \backslash G / K$, have been studied through representation theory by Matsushima, Hotta-Wallach, Borel-Wallach, Zuckerman and Casselman-Schmid. In particular if r is the real rank of G and if $1 \leq p < r$, their results show that the (o, p) Betti number must be zero. Combining this with our observations in this section, we should expect that if $1 \leq p < r$, there does not exist any parabolic subalgebra containing the Borel subalgebra r for which p is the number of non-compact roots in the unipotent radical of q . In fact, when we set out to verify this, case by case, we see that this is always the case; occasionally (e.g. $SO^\circ(n, 2)$, $SO^*(2n)$ and the exceptionals) we even get sharper results. We list below the result of doing this exercise.

(6.5) $G = SU(m, n)$ ($m \geq n$). Real rank $= n$.

In the Dynkin diagram there are $m + n - 1$ vertices $a_1, a_2, \dots, a_{m+n-1}$ enumerated in the 'usual' way. The unique non-compact root is a_m . Let q be the maximal parabolic subalgebra defined by $(a_2, a_3, \dots, a_{m+n-1})$. The cardinality of $P_{u,n}$ (non-compact roots in the unipotent radical of q) is n . There is no parabolic subalgebra q containing r for which $1 \leq \# P_{u,n} < n$.

(6.6) $G = SO^*(2n)$ ($n > 3$). Real rank $= r = [\frac{1}{2}n]$.

The Dynkin diagram has vertices $a_1, \dots, a_n$ with a_n and a_{n-1} forming a wedge at a_{n-2} . The unique non-compact root is a_n . Let q be the maximal parabolic subalgebra defined by $(a_2, \dots, a_{n-1}, a_n)$. Then the cardinality of $P_{u,n}$ is $n - 1$. There is no other q containing r for which, $1 \leq \# P_{u,n} \leq n - 1$. The (o, p) Betti numbers in this case vanish for $1 \leq p < n - 1$, even though real rank is $[\frac{1}{2}n]$.

(6.7) $G = SO(n, 2)$ ($n > 2$). Real rank $= 2$. Let $(a_1, a_2, \dots)$ be the vertices in the Dynkin diagram. Any possible wedge (which only occurs if n is even) is supposed to be at the right end. The unique non-compact simple root is a_1 . Let q be the maximal parabolic subalgebra defined by omitting the last simple root. Then cardinality of $P_{u,n} = [(n + 1)/2]$, the integral part of $[(n + 1)/2]$. There is no parabolic subalgebra q containing r for which $1 \leq \# P_{u,n} < [(n + 1)/2]$. Hence, the (o, p) Betti numbers vanish for $1 \leq p < [(n + 1)/2]$.

(6.8) $G = Sp(n, R)$. Real rank $= n$. The vertices in the Dynkin diagram are $(a_1, a_2, \dots, a_n)$ and a_n is the unique non-compact simple root. Let q be the

maximal parabolic subalgebra of g defined by $(a_2, \dots, a_n)$. Then cardinality of $P_{u,n}$ is n . There is no other parabolic subalgebra q containing r for which $1 \leq \# P_{u,n} \leq n$. Therefore, in this case, we do not get vanishing of (o, p) Betti numbers sharper than those already known.

(6.9) $G =$ the unique real form of E_6 , whose symmetric space is hermitian. The real rank is 2. The dimension of p_+ is 16. The Dynkin diagram has vertices $(a_1, a_2, a_3, a_4, a_5, a_6)$ where the part $(a_1, a_2, a_3, a_4, a_5)$ is of type A_5 and a_6 is connected to a_3 . The unique non-compact simple root is a_1 . Let q be the parabolic subalgebra of g defined by omitting a_5 . The cardinality of $P_{u,n}$ is 8. There is no other parabolic subalgebra q containing r of g for which $1 \leq \# P_{u,n} \leq 8$. Thus, in this case the (o, p) Betti numbers vanish for $1 \leq p < 8$ (even though real rank is 2). As q varies the set of numbers $\# P_{u,n}$ that we get is precisely $(0, 8, 11, 12, 13, 14, 15, 16)$. Thus, the (o, p) Betti numbers vanish also for $p = 9$ and $p = 10$.

(6.10) $G =$ the unique real form of E_7 , whose symmetric space is hermitian. The real rank is 3. The dimension of p_+ is 27. The set of numbers $\# P_{u,n}$ as q (containing r) varies is precisely $(0, 17, 21, 22, 23, 24, 25, 26, 27)$. Thus the (o, p) Betti numbers vanish for $1 \leq p < 17$ and for $p = 18, 19, 20$.

These cases cover all irreducible hermitian symmetric spaces.

(6.11) *Remark.* In the case of $G = Sp(n, R)$, the numbers $\# P_{u,n}$ as q varies consist precisely of the set $\{0\} \cup \{n + (n-1) + \dots + (n-i) \mid i = 0, 1, 2, \dots, n-1\}$. Thus if p does not belong to this set the (o, p) Betti number is zero. In the case of $G = SU(m, n)$, the set of numbers $\# P_{u,n}$ is precisely $\{mn - m'n' \mid 0 \leq m' \leq m, 0 \leq n' \leq n, m' \text{ and } n' \text{ are integers}\}$ and so the (o, p) Betti numbers vanish if p is not in this set. Similar descriptions can be obtained for the other cases also.



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