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Index

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Inventiones mathematicae

On the Canonical k-Types in the Irreducible Unitary g-Modules with Non-Zero Relative Cohomology

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1. Introduction

Let G be a connected real semi-simple Lie group with finite center, and K a maximal compact subgroup of G. We denote by g_0 and k_0 the Lie algebras of G and K respectively. $\mathfrak{g}$ and $\mathfrak{k}$ stand for their respective complexifications. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition and θ the corresponding Cartan involution. Let $(\pi, \bar{H})$ be an irreducible unitary representation of G on $\bar{H}$. Let H denote the space of K-finite vectors in $\bar{H}$. It is of current interest to study $H(\mathfrak{g}, \mathfrak{k}: H)$ the relative $(\mathfrak{g}, \mathfrak{k})$-cohomology of the $\mathfrak{g}$-module H. For this cohomology to be non-zero there is a necessary condition that the infinitesimal character of H coincides with the infinitesimal character of the trivial one dimensional representation of $\mathfrak{g}$. If this condition is satisfied then one knows $H^i(\mathfrak{g}, \mathfrak{k}: H) = \mathrm{Hom}_{\mathfrak{k}}(\Lambda^i \mathfrak{p}, H)$. In particular an irreducible $\mathfrak{k}$-component of $\Lambda^i \mathfrak{p}$ occurs in H. The precise qualitative nature of such irreducible components is not known in general (see Remark below).

The same problem arises in another context also. Let Γ be a discrete subgroup of G such that $\Gamma | G/K$ is a compact manifold. Then one knows that G acts on $L^2(\Gamma | G)$ via right regular representation and that as G-representation spaces,

$$L^2(\Gamma | G) = \bigoplus_{\pi \in \hat{G}} N_\Gamma(\pi)\pi, \qquad \text{where } N_\Gamma(\pi) = 0$$

except for countably many $\pi \in \hat{G}$ and $N_\Gamma(\pi)$ is finite for all $\pi \in \hat{G}$. Here $\hat{G}$ is the set of equivalence classes of irreducible unitary representations of G. Matsushima's formula is then given as follows:

$$\dim H^i(\Gamma | G/K) = \sum_{\substack{\pi \in \hat{G} \\ \chi_\pi = \chi_0}} N_\Gamma(\pi) \dim \mathrm{Hom}_{\mathfrak{k}}(\Lambda^i \mathfrak{p}, H)$$

Here χ_0 stands for the infinitesimal character of the trivial one dimensional representation of G. Thus we are again led to consider $\mathrm{Hom}_{\mathfrak{k}}(\Lambda^i \mathfrak{p}, H)$, for H an irreducible $\mathfrak{g}$-unitary module with $\chi_H = \chi_0$.

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Let us forget about i and have a closer look at the $\mathfrak{k}$-module structure of $\Lambda\mathfrak{p}$. If we can characterise those irreducible $\mathfrak{k}$-components of $\Lambda\mathfrak{p}$ that can occur in irreducible unitary $\mathfrak{g}$-modules H with $\chi_H=\chi_0$, such a characterisation will give a better understanding of the Betti numbers of the manifold $\Gamma|G/K$. Our main result (Theorem 1) gives the most precise information about such irreducible components of $\Lambda\mathfrak{p}$.

For stating Theorem 1 we need some notations.

Let $\mathfrak{h}_k$ be a maximal abelian subalgebra of $\mathfrak{k}$. Let $\mathfrak{h}$ be the centraliser of $\mathfrak{h}_k$ in $\mathfrak{g}$. Then $\mathfrak{h}$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$. Let Δ denote the root system for $(\mathfrak{g},\mathfrak{h})$, and $\Delta_{\mathfrak{k}}$ denote the root system for $(\mathfrak{k},\mathfrak{h}_{\mathfrak{k}})$.

Let $P_{\mathfrak{k}}$ be any system of positive roots for the roots of $(\mathfrak{k},\mathfrak{h}_{\mathfrak{k}})$. We say that a positive system P for Δ is θ-stable if $\alpha\in P$ implies $\theta\alpha\in P$ and that P is compatible with $P_{\mathfrak{k}}$ if for every $\alpha\in P_{\mathfrak{k}}$ there is a $\beta\in P$ such that $\beta|\mathfrak{h}_{\mathfrak{k}}=\alpha$. When we simply say that P is a compatible positive system for Δ then we mean that P is θ-stable. In other words if $\mathfrak{b}=\mathfrak{b}_p$ denotes the Borel subalgebra corresponding to P (see below), then $\mathfrak{b}=(\mathfrak{b}\cap\mathfrak{k})+(\mathfrak{b}\cap\mathfrak{p})$ and $(\mathfrak{b}\cap\mathfrak{k})$ is a Borel subalgebra of $\mathfrak{k}$ containing $\mathfrak{h}_k$.

Given any compatible positive system Q, we have a corresponding θ-stable Borel subalgebra $\mathfrak{b}_Q$ of $\mathfrak{g}$ given as follows:

$$\mathfrak{b}_Q=\mathfrak{h}+\sum_{\alpha\in Q}\mathfrak{g}^{\alpha}.$$

Then $\mathfrak{b}_Q\cap\mathfrak{p}$ is θ- and $\mathfrak{h}_{\mathfrak{k}}$-stable. For any such Q, we let $\delta_{Q_n}=\delta_n(Q)=\frac{1}{2}\operatorname{trace}(\mathfrak{h}_{\mathfrak{k}}|\mathfrak{b}_Q\cap p)$. We fix a positive system $P_{\mathfrak{k}}$ for $\Delta_{\mathfrak{k}}$ and a θ-stable positive system P for Δ, compatible with $P_{\mathfrak{k}}$. Let W be the Weyl group of the pair $(\mathfrak{g},\mathfrak{h})$.

We are now in a position to state our result.

Theorem 1. *Let the notations be as above.*

(a) *Let V_φ be an irreducible component of $\Lambda\mathfrak{p}$ such* that $\operatorname{Hom}_{\mathfrak{k}}(V_\varphi,H)\neq 0$ *where H is an irreducible unitary representation of G with the trivial infinitesimal character χ_0. Then there exists a θ-*stable positive system *P' compatible with $P_{\mathfrak{k}}$ and a compatible positive system $\hat{P}$ such that $\varphi=\delta_n(P')+\delta_n(\hat{P})$.*

(b) *Let H be as in a). Let P and P' be two compatible positive systems such that $\delta_n(P)+\delta_n(P')$ is $P_{\mathfrak{k}}$-dominant. (Here $P_{\mathfrak{k}}$ is the positive system for Δ_k with respect to which P is compatible). Let $\delta_n=\delta_n(P)$, $\delta'_n=\delta_n(P')$. Let $V_{\delta_n+\delta'_n}$ denote the finite dimensional irreducible $\mathfrak{k}$-module with $P_{\mathfrak{k}}$-highest weight $\delta_n+\delta'_n$. Assume that* $\operatorname{Hom}_{\mathfrak{k}}(V_{\delta_n+\delta'_n},H)\neq 0$. *Then there exists a θ-stable parabolic subalgebra $\mathfrak{q}\supseteq\mathfrak{b}_P$ with Levi decomposition* $\mathfrak{q}=\mathfrak{m}+\mathfrak{u}$, *such that $\delta_n+\delta'_n$ is the trace of $\mathfrak{h}_{\mathfrak{k}}$ on* $(\mathfrak{u}\cap\mathfrak{p})$, $\mathfrak{u}$ *being the unipotent radical of* $\mathfrak{q}$.

The proof of Theorem 1 will be given in § 2. The proof of Theorem 1 very much depends on Proposition 2.1. Using this result we first prove a) of Theorem 1. To prove b) of Theorem 1 we need a series of lemmas.

In § 3 we apply Theorem 1 to get a vanishing theorem for connected, simply connected complex simple Lie groups. This result has also been obtained by T. Enright. We also derive the vanishing theorems for $\operatorname{Sp}(n,1)$, the $\mathbb{R}$-rank 1 real form of F_4 and for the non-compact real form of G_2.

Remark. When H is a highest weight module a theorem similar to part b) of Theorem 1 was proved by R. Parthasarathy in [3]. His theorem was used by him to have a better knowledge of the Betti numbers of the type $(0, q)$ in the case of irreducible hermitian symmetric spaces. Our theorem generalises his result.

I am very grateful to Professor R. Parthasarathy for teaching me the algebraic approach to representation theory, for suggesting me this result, and for giving constant help throughout this work.

My thanks are due to the referee for his careful reading of the manuscript and suggestions concerning the organisation of the paper.

2. Proof of Theorem 1

The notations are as in § 1.

We fix a compatible positive system P for Δ and let $\delta = \delta_p$ etc.

Let L be the spin module for $SO(\mathfrak{p})$, the Lie algebra of the special orthogonal group $SO(\mathfrak{p})$. Here $\mathfrak{p}$ is endowed with the restriction of the Killing form on $\mathfrak{g}$. By composing with the adjoint action of $\mathfrak{k}$ on $\mathfrak{p}$, we obtain the spin representation σ of $\mathfrak{k}$ on L. For details see [2]. Let $(\, , \,)$ denote the Cartan-Killing form on $\mathfrak{g}$. Then $(\, , \,)$ is non-degenerate on $\mathfrak{h} \times \mathfrak{h}$ as well as on $\mathfrak{h}_{\mathfrak{k}} \times \mathfrak{h}_{\mathfrak{k}}$ under restriction. All the inner-products on $\mathfrak{h}_{\mathfrak{k}}$ will be with respect to the restriction to $\mathfrak{h}_{\mathfrak{k}}$ of the Killing form on $\mathfrak{g}$ and the Casimir element $\Omega_{\mathfrak{k}}$ of $\mathfrak{k}$ is also defined with respect to the non-degenerate restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{k}$.

The following proposition is very fundamental for us.

2.1. Proposition. *Let H be a unitary $\mathfrak{g}$-module $\mathfrak{g}$-irreducible such that the Casimir Ω of $\mathfrak{g}$ acts on H by the constant $(\mu+\delta, \mu+\delta)-(\delta, \delta)$, μ being P-integral and $P_{\mathfrak{k}}$-dominant. Assume that φ is the highest weight of an irreducible $\mathfrak{k}$-submodule of $H \otimes L$. Then we have*

$$(\varphi+\delta_{\mathfrak{k}}, \varphi+\delta_{\mathfrak{k}}) \geqq (\mu+\delta, \mu+\delta).$$

Proof. This is proved in [3; See Proposition 2.6] under the assumption rank G = rank K. But the same proof goes through in our case too word by word, since the formula for the square of the formal Dirac operator remains true even when rank G is not equal to rank K. (See 6.11, Chap. II in [1].)

We shall apply this proposition when χ_H is the trivial infinitesimal character.

We now prove part a) of Theorem 1.

Proof of Theorem 1 (a). Assume that $\mathrm{Hom}_{\mathfrak{k}}(\Lambda \mathfrak{p}, H) \neq 0$. Then since $\Lambda \mathfrak{p} = L \otimes L$ or $L \otimes L \oplus L \otimes L$, and since L is self-dual, we have $\mathrm{Hom}_{\mathfrak{k}}(L \otimes L, H) \neq 0$, and $\mathrm{Hom}_{\mathfrak{k}}(L, H \otimes L) \neq 0$. Now let V_φ be a $\mathfrak{k}$-irreducible component of $L \otimes L$ such that $V_\varphi \subseteq H$. Thus we have $\mathrm{Hom}_{\mathfrak{k}}(L \otimes L, V_\varphi) \neq 0$ and hence $\mathrm{Hom}_{\mathfrak{k}}(L, V_\varphi \otimes L) \neq 0$. Note that $V_\varphi \otimes L \subseteq H \otimes L$. Since any irreducible component of L is of the form $V_{\delta'_n}$ for some θ-stable positive system P' compatible with $P_{\mathfrak{k}}$, we have $V_{\delta'_n} \subseteq V_\varphi \otimes L$. This implies that $V_\varphi \subseteq V_{\delta'_n} \otimes L$. Hence we can write $\varphi = \delta'_n + \gamma$, where γ is a weight of the spin module L. We can write $\gamma = \delta'_n - \langle Q \rangle | \mathfrak{h}_{\mathfrak{k}}$ where $Q \subseteq P'$. To complete the proof of a) we need only prove that γ is an extreme weight of L. Let $V_{-\delta'_n}$ denote the irreducible $\mathfrak{k}$-module with lowest weight $-\delta'_n$. Then $V_{-\delta'_n}$ occurs in L. Consider $V_{\delta'_n+\gamma} \otimes L$. If $V_{[\gamma]}$ denotes the irreducible $\mathfrak{k}$-module with extreme weight γ, then by (2.26; [4]), the module $V_{[\gamma]}$ occurs in $V_{\delta'_n+\gamma} \otimes L \subseteq H \otimes L$. Therefore by Proposition 2.1, $\Omega_{\mathfrak{k}}(\gamma) \geqq |\delta|^2 - |\delta_{\mathfrak{k}}|^2$, where $\Omega_{\mathfrak{k}}(\gamma)$ denotes the scalar by which $\Omega_{\mathfrak{k}}$ acts on $V_{[\gamma]}$. But γ is a weight of L and hence is contained in $V_{\hat{\delta}_n}$ an irreducible $\mathfrak{k}$-component of L. Hence we have $\Omega_{\mathfrak{k}}(\gamma) \leqq \Omega_{\mathfrak{k}}(\hat{\delta}_n) \leqq |\delta|^2 - |\delta_{\mathfrak{k}}|^2$ and hence equality holds everywhere. In particular $\gamma = \tau \hat{\delta}_n$ where $\tau \in W_{\mathfrak{k}} \subseteq W$. ($W_{\mathfrak{k}}$ denotes the Weyl group of the pair $(\mathfrak{k}, \mathfrak{h}_{\mathfrak{k}})$.) Therefore γ is an extreme weight and $\varphi = \delta'_n + \tau \hat{\delta}_n$. This completes the proof of Theorem 1 (a).

We now turn to the proof of Theorem 1 (b). Let the notations be as in Theorem 1 (b).

2.2. Lemma. *Let* $B = \{-\alpha : \alpha \in P_{\mathfrak{k}}, (\delta'_n, \alpha) < 0\}$. *Then* $(\delta_n + \delta'_n, \beta) = 0$ *for any* $\beta \in B$.

Proof. Suppose that the lemmas is false.

Let $\alpha \in -B$ be such that $(\delta_n + \delta'_n, \alpha) \neq 0$. Since $\alpha \in -B \subseteq P_{\mathfrak{k}}$ and $\delta_n + \delta'_n$ is $P_{\mathfrak{k}}$-dominant by hypothesis, we have $(\delta_n + \delta'_n, \alpha) > 0$. We want to exhibit an irreducible $\mathfrak{k}$-submodule of $H \otimes L$, whose highest weight does not satisfy the inequality of Proposition 2.1.

Let $s_1 \in W_{\mathfrak{k}}$ be such that $s_1 \tau = -\delta'_n$, where τ is $P_{\mathfrak{k}}$-dominant integral. Let $\varphi = \delta_n + \delta'_n$. For any $s \in W_{\mathfrak{k}}$, let $V_{\varphi + s\tau}$ denote the unique irreducible finite dimensional representation of k whose highest weight lies in the $W_{\mathfrak{k}}$-orbit of $\varphi + s\tau$. Let $\Omega_{\mathfrak{k}}$ denote the Casimir element of $\mathfrak{k}$. Let C_s denote the constant by which $\Omega_{\mathfrak{k}}$ acts on $V_{\varphi + s\tau}$. For $s_2 = s_\alpha s_1$, we claim

$$C_{s_2} < C_{s_1} \ldots \tag{A}$$

The claim (A) follows if we prove that $\delta_n + \delta'_n + s_\alpha \cdot s_1 \cdot \tau$ is a weight of the irreducible module $V_{\delta_n + \delta'_n + s_1\tau} = V_{\delta_n}$, but not an extreme weight. Because then Kostant's lemma (6.8, Chap. II in [1]) gives (A).

This follows from the computations below:

$$\begin{aligned} s_\alpha(\varphi + s_1\tau) &= s_\alpha \varphi + s_2 \tau \\ &= \varphi + s_2\tau - \frac{2(\varphi, \alpha)}{(\alpha, \alpha)} \alpha \ldots \end{aligned} \tag{B}$$

Also

$$s_\alpha(\varphi + s_1\tau) = \varphi + s_1\tau - \frac{2(\varphi, \alpha)}{(\alpha, \alpha)}\alpha - \frac{2(s_1\tau, \alpha)}{(\alpha, \alpha)}\alpha \ldots \tag{C}$$

From (B) and (C) it follows that

$$\varphi + s_2\tau = \varphi + s_1\tau - \frac{2(s_1\tau, \alpha)}{(\alpha, \alpha)}\alpha.$$

By hypothesis on α,

$$\frac{2(s_1\tau, \alpha)}{(\alpha, \alpha)} = \frac{2(-\delta'_n, \alpha)}{(\alpha, \alpha)} > 0$$

and

$$\frac{2(\varphi, \alpha)}{(\alpha, \alpha)} = 2\frac{(\delta_n + \delta'_n, \alpha)}{(\alpha, \alpha)} > 0 \ldots . \tag{D}$$

Now both $\varphi+s_1\tau$ and $s_\alpha(\varphi+s_1\tau)$ are weights of $V_{\varphi+s_1\tau}$. Hence we see that $\varphi+s_2\tau$ lies in between $s_\alpha(\varphi+s_1\tau)$ and $(\varphi+s_1\tau)$, in the α-string of weights of $V_{\varphi+s_1\tau}$ through $\varphi+s_1\tau$. Since the α-string of weights through a given weight of an irreducible module is unbroken, we see that $\varphi+s_2\tau$ is a weight of $V_{\varphi+s_1\tau}$ and by (D) above, it is not an extreme weight. Thus we have proved (A).

If t denotes the unique element of $W_{\mathfrak{k}}$ such that $tP_{\mathfrak{k}}=-P_{\mathfrak{k}}$, then we can go from s_2 to t by means of $P_{\mathfrak{k}}$-simple reflections: $t=w_r\ldots w_1 s_2$, where each w_i is a $P_{\mathfrak{k}}$-simple reflection and where the above expression for t is 'minimal' i.e. $l(t)=r+l(s_2)$. Here $l(s)$, for $s\in W_{\mathfrak{k}}$, denotes the length of s. By an argument similar to one used above, we can show that

$$C_t\leqq C_{w_{r-1}\ldots w_1 s_2}\leqq C_{w_1 s_2}\leqq C_{s_2}<C_{s_1}.$$

By (2.26, [4]), $V_{\varphi+t\tau}$ occurs in $V_\varphi\otimes V_\tau$.

Now $V_\varphi=V_{\delta_n+\delta_n'}\subseteq H$ by hypothesis and $V_\tau\subseteq L$ by assumption on τ. Hence we see that $V_{\varphi+t\tau}$ occurs in $V_{\delta_n+\delta_n'}\otimes V_\tau\subseteq H\otimes L$.

But $C_{s_1}=(\delta,\delta)-(\delta_{\mathfrak{k}},\delta_{\mathfrak{k}})$, since $C_{s_1}=\Omega_{\mathfrak{k}}$ on $V_{\delta_n+\delta_n'+s_1\tau}=V_{\delta_n+\delta_n'-\delta_n'}=V_{\delta_n}\subseteq L$, and V_{δ_n} is an irreducible component of L on which Ω_k acts by the scalar $(\delta,\delta)-(\delta_{\mathfrak{k}},\delta_{\mathfrak{k}})$.

Let w be an element of $W_{\mathfrak{k}}$ such that $w(\delta_n+\delta_n'+t\tau)$ is $P_{\mathfrak{k}}$-dominant. Then we have

$$C_t=(w(\delta_n+\delta_n'+t\tau)+\delta_k,\ w(\delta_n+\delta_n'+t\tau)+\delta_{\mathfrak{k}})-(\delta_{\mathfrak{k}},\delta_{\mathfrak{k}})$$

since $C_t<C_{s_1}$ we have

$$(w(\delta_n+\delta_n'+t\tau)+\delta_{\mathfrak{k}},\ w(\delta_n+\delta_n'+t\tau)+\delta_{\mathfrak{k}})-(\delta_{\mathfrak{k}},\delta_{\mathfrak{k}})$$
$$<(\delta,\delta)-(\delta_{\mathfrak{k}},\delta_{\mathfrak{k}}).$$

Hence

$$(w(\delta_n+\delta_n'+t\tau)+\delta_{\mathfrak{k}},\ w(\delta_n+\delta_n'+t\tau)+\delta_{\mathfrak{k}})$$
$$<(\delta,\delta)\ldots \tag{E}$$

Since the representation $V_{w(\delta_n+\delta_n'+t\tau)}$ occurs in $H\otimes L$, (E) contradicts the inequality established in Proposition 2.1. This contradiction shows that we have $(\delta_n+\delta_n',\alpha)=0$ for all $\alpha\in -B$.

This completes the proof of the Lemma.

2.3. Lemma. *We set* $A_1=\{\alpha\in P_{\mathfrak{k}}\colon (\delta_n+\delta_n',\alpha)>0\}$ *and*

$$B_1=\{-\alpha\colon \alpha\in P_{\mathfrak{k}},\ (\delta_n+\delta_n',\alpha)=0\},$$

and $P_{\mathfrak{k}}''=A_1\cup B_1$.

Then i) $P_{\mathfrak{k}}''$ *is a positive system for* $\Delta_{\mathfrak{k}}$ *ii)* $\delta_n+\delta_n'$ *is* $P_{\mathfrak{k}}''$*-dominant and iii)* δ_n' *is* $P_{\mathfrak{k}}''$*-dominant.*

Proof. Recall that by hypothesis $\delta_n+\delta_n'$ is $P_{\mathfrak{k}}$-dominant. Now to any dominant $\lambda\in\mathfrak{h}_{\mathfrak{k}}^*$, we can associate a parabolic subalgebra $\mathfrak{q}=\mathfrak{m}+\mathfrak{u}=h+\sum_{\alpha\in P_{\mathfrak{k}}} k^\alpha$ of $\mathfrak{k}$ such that

$$P_{\mathfrak{k},\mathfrak{m}}=\{\alpha\in P_{\mathfrak{k}}\colon (\lambda,\alpha)=0\}$$

and

$$P_{\mathfrak{k},\mathfrak{u}}=\{\alpha\in P_{\mathfrak{k}}\colon (\lambda,\alpha)>0\}.$$

Here $P_{\mathfrak{k},\mathfrak{m}}$ denotes those roots $\alpha \in P_{\mathfrak{k}}$ such that $\mathfrak{k}^{\alpha} \subseteq \mathfrak{m}$, the reductive part of $\mathfrak{q}$ etc. This parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{k}$ defines a positive system $P_{\mathfrak{k}}^{-}$ for $\Delta_{\mathfrak{k}}$ given as follows:

$$P_{\mathfrak{k}}^{-} = (-P_{\mathfrak{k},\mathfrak{m}}) \cup P_{\mathfrak{k},u}.$$

If we take λ to be $\delta_n + \delta'_n$ we see that $P''_{\mathfrak{k}}$ is precisely the positive system associated to by the $\mathfrak{k}$-parabolic subalgebra corresponding to $\delta_n + \delta'_n$. Thus (i) is proved ii) is obvious.

To prove iii) Let $\alpha \in A_1$ be such that $(\delta'_n, \alpha) < 0$. Then $-\alpha \in B$ and hence by Lemma 2.2 $(\delta_n + \delta'_n, \alpha) = 0$ which is impossible sicne $\alpha \in A_1$. Let $\beta \in B_1$ be such that $(\delta'_n, \beta) < 0$. Then $\beta = -\alpha$ for some $\alpha \in P_{\mathfrak{k}}$ and $(\delta_n + \delta'_n, \alpha) = 0$.

So $(\delta_n, \alpha) = -(\delta'_n, \alpha) < 0$. But $(\delta_n, \alpha) > 0$, since $\alpha \in P_{\mathfrak{k}}$. Hence δ'_n is $P''_{\mathfrak{k}}$-dominant.

Thus the lemma is completely proved.

We recall that $W_{\mathfrak{k}}$ the Weyl group of the pair $(\mathfrak{k}, \mathfrak{h}_{\mathfrak{k}})$ is canonically imbedded in W the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$. For any positive system Q, we denote by $\mathfrak{b}(Q)$ the corresponding Borel subalgebra. If Q is a θ-stable compatible positive system of roots, then we have

$$\mathfrak{b}(Q) = (\mathfrak{b}(Q) \cap \mathfrak{k}) \oplus (\mathfrak{b}(Q) \cap \mathfrak{p}),$$

with $\mathfrak{b}(Q) \cap \mathfrak{k} = \mathfrak{b}(Q_{\mathfrak{k}})$ the Borel subalgebra of $\mathfrak{k}$ corresponding to $Q_{\mathfrak{k}}$ (see [1]). Since P and P' are compatible positive system of roots, they give rise to θ-stable parabolic subalgebras

$$\mathfrak{b}(P) = \mathfrak{b} = \mathfrak{b}_{\mathfrak{k}} + (\mathfrak{b} \cap \mathfrak{p}),$$

$$\mathfrak{b}(P') = \mathfrak{b}' = \mathfrak{b}'_{\mathfrak{k}} + (\mathfrak{b}' \cap \mathfrak{p}).$$

2.4. Lemma. Set $\mathfrak{b}'' = \mathfrak{b}''_{\mathfrak{k}} + (\mathfrak{b}' \cap \mathfrak{p})$, where $\mathfrak{b}''_{\mathfrak{k}}$ is the Borel subalgebra of $\mathfrak{k}$ corresponding to $P''_{\mathfrak{k}}$. Then $\mathfrak{b}''$ is the θ-stable Borel subalgebra of $\mathfrak{g}$ corresponding to a θ-stable positive system P'' of Δ, which is compatible with $P''_{\mathfrak{k}}$.

Furthermore, $\delta_{\mathfrak{k}} + \delta''_{\mathfrak{k}}$ is 0 on B_1 (and hence $\delta + \delta''$ is 0 on B_1).

Proof. If P' is any compatible positive system such that $\delta_n(P') = \delta'_n$ is $P''_{\mathfrak{k}}$-dominant, we claim that $\mathfrak{b}'' = \mathfrak{b}''_{\mathfrak{k}} + (\mathfrak{b}' \cap \mathfrak{p})$ is a θ-stable Borel subalgebra corresponding to a compatible positive system for Δ. This we prove by downward induction on $|P'_{\mathfrak{k}} \cap P''_{\mathfrak{k}}|$. If $|P'_{\mathfrak{k}} \cap P''_{\mathfrak{k}}| = |P''_{\mathfrak{k}}|$, then we are through. We assume by way of induction that for any compatible positive system P' of with $|P'_{\mathfrak{k}} \cap P''_{\mathfrak{k}}| > i$ with $i + 1 \leqq |P''_{\mathfrak{k}}|$ and such that δ'_n is $P''_{\mathfrak{k}}$-dominant, we have shown that b'' is a θ-stable parabolic subalgebra of $\mathfrak{g}$.

Let now $P'_{\mathfrak{k}}$ be such that $|P'_{\mathfrak{k}} \cap P''_{\mathfrak{k}}| = i$. Hence there exists a $P'_{\mathfrak{k}}$-simple root $\alpha \in P'_{\mathfrak{k}} \cap -P''_{\mathfrak{k}}$. Since δ'_n is $P'_{\mathfrak{k}}$ as well as $P''_{\mathfrak{k}}$-dominant, we have $(\delta'_n, \alpha) = 0$. We claim that if $\beta \in P'$ is such that $\beta | \mathfrak{h}_{\mathfrak{k}} = \alpha$, then β is supported on $\mathfrak{h}_{\mathfrak{k}}$ and β is P'-simple.

Suppose that β were not supported on $\mathfrak{h}_{\mathfrak{k}}$. Then $(\beta, \beta) > (\alpha, \alpha)$. We therefore have

$$\frac{2(\delta', \beta)}{(\beta, \beta)} = \frac{2(\delta'_{\mathfrak{k}} + \delta'_n, \beta)}{(\beta, \beta)} = \frac{2(\delta'_{\mathfrak{k}}, \alpha)}{(\beta, \beta)} + \frac{2(\delta'_n, \alpha)}{(\beta, \beta)}$$

(since $\delta'_{\mathfrak{k}}$ and δ'_n are supported on $\mathfrak{h}_{\mathfrak{k}}$)

$$= \frac{2(\delta'_{\mathfrak{k}}, \alpha)}{(\beta, \beta)} < \frac{2(\delta'_{\mathfrak{k}}, \alpha)}{(\alpha, \alpha)} = 1.$$

This contradiction shows that β is supported on $\mathfrak{h}_{\mathfrak{k}}$ and that β is P'-simple.

Now we set $P^1 = s_\alpha P'$. Then P^1 is a compatible positive system, since $\alpha \in P'_{\mathfrak{k}}$ and $\mathfrak{b}(P^1) \cap \mathfrak{p} = \mathfrak{b}' \cap \mathfrak{p}$. If $\delta_n^1 = \delta_n(P^1)$, then we have $\delta_n^1 = \delta'_n$. We also see that $|P_{\mathfrak{k}}^1 \cap P''_{\mathfrak{k}}| = i+1$. Hence by induction hypothesis $\mathfrak{b}'' = \mathfrak{b}''_{\mathfrak{k}} + (\mathfrak{b}^1 \cap \mathfrak{p}) = \mathfrak{b}''_{\mathfrak{k}} + (\mathfrak{b}' \cap \mathfrak{p})$ is a θ-stable Borel subalgebra of $\mathfrak{g}$.

It remains to prove that $\delta_{\mathfrak{k}} + \delta''_{\mathfrak{k}}$ is 0 on B_1. But this is clear, since in the notation of 2.3, we have $\delta_{\mathfrak{k}} + \delta''_{\mathfrak{k}} =$ trace of $\mathfrak{h}_{\mathfrak{k}}$ on $\mathfrak{u}$ and $B_1 = P_{\mathfrak{m},\mathfrak{k}}$.

This completes the proof of Lemma 2.4.

Let $\delta'' = \delta(P'')$, half the sum of the roots in P'', P'' being as in 2.4 above.

2.5. Lemma. $\delta + \delta''$ is P-dominant.

Proof. Suppose that $\delta + \delta''$ is not P-dominant. Then there exists a P-simple root α such that $(\delta + \delta'', \alpha) < 0$. Notice that $\alpha \in -P''$, since otherwise, $\alpha \in P''$ and hence $(\delta + \delta'', \alpha) = (\delta, \alpha) + (\delta'', \alpha)$ and each of these terms is positive and hence $(\delta + \delta'', \alpha)$ is positive, contradicting our assumption on α. Hence $-\alpha \in P''$.

We remark that it is enough to prove that $-\alpha$ is simple in P''. For then $\dfrac{2(\delta'', \alpha)}{(\alpha, \alpha)} = -1$ and since α is simple in $P\, \dfrac{2(\delta, \alpha)}{(\alpha, \alpha)} = 1$. Thus we get $(\delta + \delta'', \alpha) = 0$. This contradiction shows that no such α can exist, in other words $\delta + \delta''$ is P-dominant.

That $-\alpha$ is P''-simple follows if we establish (*):

$$(\delta'' + \alpha, \delta'' + \alpha) = (\delta'', \delta'') \tag{*}$$

For then by expanding, we have

$$(\delta'', \delta'') + 2(\delta'', \alpha) + (\alpha, \alpha) = (\delta'', \delta'').$$

This implies that $2(\delta'', \alpha) + (\alpha, \alpha) = 0$ i.e. $2(\delta'', \alpha)/(\alpha, \alpha) = -1$.

We set out to prove (*). We first observe that such an α is θ-stable i.e. $\theta\alpha = \alpha$. In fact, we claim $\mathfrak{g}^\alpha \subseteq \mathfrak{p}$. Suppose that $\mathfrak{g}^\alpha \nsubseteq \mathfrak{p}$. We may as well assume $\theta\alpha \neq \alpha$, since otherwise $\mathfrak{g}^\alpha \subseteq \mathfrak{k}$ and hence $(\delta + \delta'', \alpha) > 0$. This contradicts our assumption on α. Hence we may assume $\theta\alpha \neq \alpha$.

Let $\mathfrak{h}_{\mathfrak{p}}^*$ stand for the orthogonal complement of $\mathfrak{h}_{\mathfrak{k}}^*$ in $\mathfrak{h}^*$ with respect to the restriction of the Killing form on $\mathfrak{g}^*$. We write $\alpha = (\alpha_1, \alpha_2) \in \mathfrak{h}_{\mathfrak{k}}^* + \mathfrak{h}_{\mathfrak{p}}^*$. Since P and P'' are θ-stable, $\delta \in \mathfrak{h}_{\mathfrak{k}}^*$ and $\delta'' \in \mathfrak{h}_{\mathfrak{k}}^*$. Therefore $(\delta + \delta'', \alpha) = (\delta + \delta'', \alpha_1)_{\mathfrak{h}_{\mathfrak{k}}}$ and hence $(\delta + \delta'', \alpha_1)_{\mathfrak{h}_{\mathfrak{k}}} < 0$. If $X \in \mathfrak{g}$ is of weight α for $\mathfrak{h}$ then α_1 is the $\mathfrak{h}_{\mathfrak{k}}$-weight of $X + \theta X$. But then $X + \theta X$ is such that $\theta(X + \theta X) = X + \theta X$ and hence lies in $\mathfrak{k}$. Now $X \in \mathfrak{b}$, the θ-stable Borel subalgebra corresponding to the positive system P and so $\theta X \in \mathfrak{b}$ and hence $X + \theta X \in \mathfrak{b} \cap \mathfrak{k} = \mathfrak{b}_{\mathfrak{k}}$. In otherwords, we have shown that $\alpha_1 \in P_{\mathfrak{k}}$. But then as we saw above, $(\delta + \delta'', \alpha_1)_{\mathfrak{h}_{\mathfrak{k}}} > 0$. This contradiction shows that $\mathfrak{g}^\alpha \subseteq \mathfrak{p}$.

Let $\tilde{P} = s_\alpha P$. Since $\theta\alpha = \alpha$, and $\mathfrak{g}^\alpha \subseteq \mathfrak{p}$, $\tilde{P}$ is θ-stable and compatible with $P_{\mathfrak{k}}$. Since α is P-simple, we have $\tilde{\delta}_n = \delta_n - \alpha_1$. We define $\tilde{\delta}''_n = \delta''_n + \alpha_1$. We claim that $\tilde{\delta}''_n$ is an extreme weight of the spin module L. To see this we note that $\tilde{\delta}_n + \tilde{\delta}''_n = \delta_n + \delta''_n$ is $P_{\mathfrak{k}}$-dominant and is such that the corresponding $\mathfrak{k}$-irreducible module $V_{\tilde{\delta} + \tilde{\delta}''_n}$ occurs in H. By an argument similar to the one employed in the proof of Theorem 1(a), we can show that $\tilde{\delta}''_n$ has to be an extreme weight of L. That is, there exists a compatible positive system $\tilde{P}''$ such that $\delta_n(\tilde{P}'') = \tilde{\delta}''_n$.

We can replace (δ_n, δ'_n) by $(\tilde{\delta}_n, \tilde{\delta}''_n)$ in Lemmas 2.3 and 2.4 to conclude that there exists a positive system $P_{\mathfrak{k}}^1$ for $\Delta_{\mathfrak{k}}$ and that $\mathfrak{b}_{\mathfrak{k}}^1 \oplus (\mathfrak{b}'' \cap \mathfrak{p})$ is the Borel subalgebra corresponding to a compatible positive system P^1 of Δ. Since $\tilde{\delta}_n + \tilde{\delta}''_n = \delta_n + \delta''_n = \delta_n + \delta'_n$, we have $P_{\mathfrak{k}}^1 = P_{\mathfrak{k}}''$ and hence $\delta_{\mathfrak{k}}^1 = \delta_{\mathfrak{k}}''$. The Casimir $\Omega_{\mathfrak{k}}$ of $\mathfrak{k}$ acts on the extreme weight spaces of L corresponding to δ_n^1 and δ_n'' by the scalars $(\delta^1, \delta^1) - (\delta_{\mathfrak{k}}^1, \delta_{\mathfrak{k}}^1)$ and $(\delta'', \delta'') - (\delta_{\mathfrak{k}}'', \delta_{\mathfrak{k}}'')$ respectively. But these two scalars are the same and $\delta_{\mathfrak{k}}^1 = \delta_{\mathfrak{k}}''$. Therefore we have $(\delta^1, \delta^1) = (\delta'', \delta'')$. Since $\delta^1 = \delta'' + \alpha$, we see that we have established (*).

This completes the proof of the Lemma 2.5.

We can now interchange the roles of P and P'' and see that $\delta + \delta''$ is P''-dominant too. (Note that in this case $(P_k'')'' = P_{\mathfrak{k}}$ and hence the new positive system we attach to δ_n is P itself.)

Proof of Theorem 1(b). We set $P_{\mathfrak{m}} = \{\alpha \in P : (\delta + \delta'', \alpha) = 0\}$ and $P_{\mathfrak{u}} = \{\alpha \in P : (\delta + \delta'', \alpha) > 0\}$. Since $\delta + \delta''$ is dominant with respect to P as well as P'', we see that $P_{\mathfrak{m}} = P \cap -P''$ and $P_{\mathfrak{u}} = P \cap P''$. We define $\mathfrak{m} = \mathfrak{h} + \sum_{+\alpha \in P_{\mathfrak{m}}} \mathfrak{g}^{\alpha} + \sum_{\alpha \in P_{\mathfrak{m}}} \mathfrak{g}^{-\alpha}$, and $\mathfrak{u} = \sum_{\alpha \in P_{\mathfrak{u}}} \mathfrak{g}^{\alpha}$. Then $\mathfrak{q} = \mathfrak{m} + \mathfrak{u}$ is a θ-stable parabolic subalgebra containing the Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \sum_{\alpha \in P} \mathfrak{g}^{\alpha}$. It is clear that $\delta_n + \delta''_n = \delta_n + \delta'_n = 2\delta_{\mathfrak{q},n} =$ the trace of $\mathfrak{h}_{\mathfrak{k}}$ on $\mathfrak{u} \cap \mathfrak{p}$.

Thus the theorem is completely proved.

3. Applications: Vanishing theorems

3.1. In this section we show how our main result can be used to derive vanishing theorems for the relative cohomology groups of complex simple Lie groups, Sp(n. 1), the real rank-1 form of F_4 and the unique non-compact real form of G_2. In fact, it is our belief that the main result gives the 'philosophy' behind the 'generic' vanishing theorems.

3.2. Let $\mathfrak{g}$, $\mathfrak{k}$, P etc. be as usual. Let $\mathfrak{q}$ be any θ-stable parabolic subalgebra (containing, of course, the Borel subalgebra $\mathfrak{b}(P) = \mathfrak{b}$). Let $\mathfrak{q} = \mathfrak{m} + \mathfrak{u}$ be its Levi decomposition. Let $X_{\gamma_1}, \ldots, X_{\gamma_n}$ be a basis of $\mathfrak{u} \cap \mathfrak{p}$, where γ_i is the $\mathfrak{h}_{\mathfrak{k}}$-weight of X_{γ_i}. Then $X_{\gamma_1} \wedge \ldots \wedge X_{\gamma_\pi} \in \Lambda^n \mathfrak{p}$ and is of weight $2\delta_{\mathfrak{q},n}$. It is also a $P_{\mathfrak{k}}$-extreme vector. For if $\alpha \in P_{\mathfrak{k}}$, then

$$X_\alpha \cdot (X_{\gamma_1} \wedge \ldots \wedge X_{\gamma_\pi}) = \sum_i (X_{\gamma_1} \wedge \ldots \wedge [X_\alpha, X_{\gamma_i}] \wedge X_{\gamma_{i+1}} \wedge \ldots \wedge X_{\gamma_\pi}).$$

Since $[X_\alpha, X_{\gamma_i}]$ lies in $\mathfrak{u}$ either $[X_\alpha, X]$ is 0 or is of the form cX_j, for $j \neq i$, for some non zero constant c. Hence the assertion.

3.3. Here we set up the notation for complex simple Lie groups. Let G be a connected, simply connected complex simple Lie group and let $\mathfrak{g}_0$ denote the Lie algebra of G. For any complex vector space V, we denote by $V_{\mathbb{R}}$ the underlying real vector space. Let $\mathfrak{k}_0$ be a compact real form of $\mathfrak{g}_0$. Let J denote the complex structure of $\mathfrak{g}_0$. Then $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ with $\mathfrak{p}_0 = J\mathfrak{k}_0$, is Gartan decomposition for $\mathfrak{g}_0$. We denote by θ the corresponding Cartan involution. Let $\mathfrak{t}_0$ be a Cartan subalgebra of $\mathfrak{k}_0$. If we put $\mathfrak{a}_0 = J\mathfrak{t}_0$, then $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$.

For $X + JY \in \mathfrak{k}_0 + J\mathfrak{k}_0 = \mathfrak{g}_0$, we define the conjugation $\overline{X + JY} = X - JY$. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_{0,\mathbb{R}}$. We define i: $\mathfrak{g}_{0,\mathbb{R}} \to \mathfrak{g}_0 \times \mathfrak{g}_0$ by setting $i(X) = (X, \bar{X})$, for $X \in \mathfrak{g}_{0,\mathbb{R}}$. Then i is an injection of $\mathfrak{g}_{0,\mathbb{R}}$ into $\mathfrak{g}_0 \times \mathfrak{g}_0$ and via this map we identify $\mathfrak{g}$ with $\mathfrak{g}_0 \times \mathfrak{g}_0$. Thus $(\mathfrak{g}_0 \times \mathfrak{g}_0, i)$ is a complexification of $\mathfrak{g}_{0,\mathbb{R}}$.

We set

$$\mathfrak{k} = \{(X, X) \colon X \in \mathfrak{g}_0\}$$

and

$$\mathfrak{p} = \{(X, -X) \colon X \in \mathfrak{g}_0\}.$$

Then $(\mathfrak{k}, i|\mathfrak{k}_0)$ is a complexification of $\mathfrak{k}_0$ etc. We set $\mathfrak{h} = \mathfrak{h}_0 \times \mathfrak{h}_0$. We let $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$. Then $\mathfrak{t}$ and $\mathfrak{a}$ are the complexifications of $\mathfrak{t}_0$ and $\mathfrak{a}_0$ respectively.

Let Δ_0 denote the set of roots of $(\mathfrak{g}_0, \mathfrak{h}_0)$ and let P_0 be a positive system for Δ_0. For λ, $\mu \in \mathfrak{h}_0^*$, we let (λ, μ) denote the element in $\mathfrak{h}^*$ defined by $(\lambda, \mu)(H_1, H_2) = \lambda(H_1) + \mu(H_2)$ for $H_1, H_2 \in \mathfrak{h}_0$. Using this we see that $\Delta = (\Delta_0 \times \{0\}) \cup (\{0\} \times \Delta_0)$ is the root system for $(\mathfrak{g}, \mathfrak{h})$ and $P = (P_0 \times \{0\}) \cup (\{0\} \times P_0)$ is a positive system for Δ. It is easily seen that P is a compatible postiive system for Δ.

3.4. If $\mathfrak{b}_0 = \mathfrak{h}_0 + \sum_{\alpha \in P_0} \mathfrak{g}_0^\alpha$ then $\mathfrak{b} = \mathfrak{b}_0 \times \mathfrak{b}_0$ is the θ-stable Borel subalgebra $\mathfrak{b}(P) = \mathfrak{b}$ corresponding to P. Any θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ containing $\mathfrak{b}$ is of the form $\mathfrak{q}_0 \times \mathfrak{q}_0$, where $\mathfrak{q}_0$ is a parabolic subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{b}_0$. Let $\mathfrak{q} = \mathfrak{m} + \mathfrak{u}$ and $\mathfrak{q}_0 = \mathfrak{m}_0 + \mathfrak{u}_0$ be the Levi decompositions of $\mathfrak{q}$ and $\mathfrak{q}_0$ respectively. Then we have $\mathfrak{u} \cap \mathfrak{p} = \{(X, -X) \colon X \in \mathfrak{u}_0\}$ is canonically isomorphic to $\mathfrak{u}_0$.

3.5. We keep the notations of 3.4. We now prove the following lemma.

Lemma. Let $R = \dim(\mathfrak{u} \cap \mathfrak{p})$. Then $2\delta_{\mathfrak{q},n}$ is not a weight of $\Lambda^i \mathfrak{p}$ for $i < R$.

Proof. By what we said above, it is enough to show that for $\mathfrak{q}_0 = \mathfrak{m}_0 + \mathfrak{u}_0$ and $2\delta_{\mathfrak{u}_0} = \sum_{\mathfrak{g}_0^\alpha \subseteq \mathfrak{u}_0} \alpha$, $2\delta_{\mathfrak{u}_0}$ is not a weight of $\Lambda^i \mathfrak{g}_0$ for $i < \dim \mathfrak{u}_0$.

We now use the bijective correspondence between the subsets of the set of simple roots and the parabolic subalgebras $\mathfrak{q}_0$ of $\mathfrak{g}_0$. Let $\alpha_1, \ldots, \alpha_l$ be the set of simple roots of P_0. Let us assume that $\mathfrak{q}_0$ corresponds to $\{\alpha_1, \ldots, \alpha_j\} j < l$. i.e. $\mathfrak{q}_0 = \mathfrak{m}_0 + \mathfrak{u}_0$ where

$$\mathfrak{m}_0 = \mathfrak{h}_0 + \sum_{\alpha \in \Delta_j} \mathfrak{g}_0^\alpha,$$

and

$$\mathfrak{u}_0 = \sum_{\alpha \in P_{0,\mathfrak{u}}} \mathfrak{g}_0^\alpha.$$

Here $\Delta_j = \{\alpha \in \Delta_0 \colon \alpha$ lies in the integral linear combinations of $\alpha_1, \ldots, \alpha_j\}$ and $P_{0,\mathfrak{u}} = P_0 \setminus \Delta_j$. For any $\gamma \in \Delta$, we let $m_r(\gamma)$ = the coefficient of α_r in the expression for γ in terms of the simple roots $\alpha_1, \ldots, \alpha_l$. If $\gamma \in P_0$, then $\gamma \in P_{0,\mathfrak{u}}$ if and only if $m_r(\gamma) > 0$ for some $r > j$. A moment's reflection now shows that if $r < R$, then $X_{\beta_1} \wedge \ldots \wedge X_{\beta_r}$ cannot be of weight $2\delta_{\mathfrak{u}_0}$. This completes the proof of the lemma.

3.6. The lemma 3.4 shows that $2\delta_{\mathfrak{q},n}$ where $\mathfrak{q}$ is a proper θ-stable parabolic subalgebra, cannot occur in $\Lambda^i \mathfrak{p}$ for $i < \dim(\mathfrak{u} \cap \mathfrak{p})$. But it may still be possible that the trivial one dimensional representation of $\mathfrak{k}$ occurs in $\Lambda^i \mathfrak{p}$ for some $i \geqq 1$ and also in an irreducible unitary representation H of G with $\chi_H = \chi_0$. The proposition below shows that this can happen only if $H = C$, the trivial representation of G.

Proposition. *Let H be an irreducible unitary $\mathfrak{g}$-module with $\chi_H = \chi_0$. Let H have a non-zero $\mathfrak{k}$-fixed vector. Then H is the trivial one dimensional representation of $\mathfrak{g}$.*

Proof. If $H^{\mathfrak{k}} = \{v \in H: X \cdot v = 0 \text{ for } X \in \mathfrak{k}\}$, then it is well-known that $H^{\mathfrak{k}}$, is one dimensional. Let $v_0 \in H^{\mathfrak{k}}$, $v_0 \neq 0$.

If H were non-trivial, since H is irreducible, we see that the $\mathfrak{k}$-module map $\mathfrak{p} \times C \cdot v_0 \to H$ given by $(X, v_0) \mapsto X \cdot v_0$ is non-zero. Since G is simple $\mathfrak{p}$ is irreducible. Hence as $\mathfrak{k}$-modules $\mathfrak{p}$ is isomorphic to $\{X \cdot v_0 : X \in \mathfrak{p}\}$. Hence we have $\mathrm{Hom}_{\mathfrak{k}}(\Lambda^1 \mathfrak{p}, H) \neq 0$. Theorem 1 now implies that the highest weight of the $\mathfrak{k}$-module $\Lambda^1 \mathfrak{p} = \mathfrak{p}$ is of the form $2\delta_{\mathfrak{q},n}$ for some θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$. By 3.4 we see that this is equivalent to saying that the largest root α of $\mathfrak{g}_0$ is of the form $2\delta_{\mathfrak{u}_0}$ for some parabolic subalgebra $\mathfrak{q}_0 = \mathfrak{m}_0 + \mathfrak{u}_0$ of $\mathfrak{g}_0$. An examination of the Dynkin diagram shows that this can happen only if $\mathfrak{g}_0 = sl(2, C)$. But in that case G is $SL(2, C)$ and the proposition is true for G. This contradiction shows that H is the trivial representation of G.

Remark. This proposition is known and a proof can be found in ([1], Ch. II cf. the proof of 8.6). I thank the referee for bringing this fact to my attention. However, I believe the above proof may be of interest to some.

3.7. Let us set $r(G) = \inf_{\mathfrak{q}_0} \{\dim \mathfrak{u}_0\}$ where $\mathfrak{q}_0$ runs through all proper parabolic subalgebras $\mathfrak{q}_0$ of $\mathfrak{g}_0$, and where $\mathfrak{u}_0$ is the unipotent radical of $\mathfrak{q}_0$.

Theorem 2. Let G be as in 3.3 and $r(G)$ as above. If H is an irreducible, unitary, non-trivial representation of G, then $H^i(\mathfrak{g}, \mathfrak{k}; H) = 0$ for $i < r(G)$ where $r(G)$ is given below:

Type	$r(G)$	Type	$r(G)$
A	l	E_6	16
B	$2l-1$	E_7	27
C	$2l-1$	E_8	57
$\vdots$	$\cdots$	F_4	15
D	$2(l-1)$	G_2	5

Proof. Follows immediately from 3.4–3.6 and by computing the dimensions of $\mathfrak{u}_0$.

Remark. T.J. Enright proves this theorem in "Relative Lie algebra cohomology and unitary representations of complex Lie groups" Duke M.J. 46 (1979), 513–525 for $H^i(\mathfrak{g}, \mathfrak{k}; F \otimes H)$ where F is a finite dimensional irreducible $\mathfrak{g}$-module.

3.7. Here we derive vanishing theorems for the groups which are locally isomorphic to one of the groups: $\mathrm{Sp}(n, 1)$, the $\mathbb{R}$-rank 1-form of F_4, the split real form of G_2.

We describe the root systems for each of these. We follow the notations of Bourbaki. In all these cases rank $G = \operatorname{rank} K$ and $\mathfrak{p}$ is irreducible as a $\mathfrak{k}$-module.

Sp(n.1). P has simple roots $\{\alpha_1, \ldots, \alpha_{n+1}\}$ and $P_\mathfrak{k}$ has simple roots $\{2\varepsilon_1; \alpha_2, \alpha_3 \ldots \alpha_{n+1}\}$. Then the $P_\mathfrak{k}$-highest weight of $\mathfrak{p}$ is $\varepsilon_1 + \varepsilon_2$.

The $\mathbb{R}$-*rank* 1 *form of* F_4. P has simple roots $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $P_\mathfrak{k}$ has simple roots $\{\varepsilon_1 - \varepsilon_2; \alpha_1, \alpha_2, \alpha_3\}$. The $P_\mathfrak{k}$ highest weight of $\mathfrak{p}$ is $\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$.

The unique non-compact real form of G_2. P has simple roots $\{\alpha_1, \alpha_2\}$ and $P_\mathfrak{k}$ has simple roots $\{\alpha_1; 3\alpha_1 + 2\alpha_2\} = P_\mathfrak{k}$ itself. The highest weight of $\mathfrak{p}$ is $3\alpha_1 + \alpha_2$.

From the description of the highest weight of $\mathfrak{p}$, and Theorem 1(b) it is clear that if $\mathfrak{g}$ is one of the above three types, then $H^1(\mathfrak{g}, \mathfrak{k}; H) = 0$, for an irreducible unitary non-trivial $\mathfrak{g}$-module H. This result is proved also in ([1], Chap. II).

Thus we see that the analogues of Lemma 3.5 and Proposition 3.6 are true in this case too. Hence we have proved the following theorem, except for $i = 2$ in (3).

Theorem 3. *Let G be locally isomorphic to one of the groups: $Sp(n, 1)$, $\mathbb{R}$-rank 1 form of F_4 and the unique non-compact real form of G_2. Let H be an irreducible unitary, non trivial representation of G. Then $H^i(\mathfrak{g}, \mathfrak{k}; H) = 0$ for $i < r(G)$ where $r(G)$ is given as follows:*

(1) *For type $Sp(n, 1)$: $r(G) = 2$.*
(2) *For type F_4: $r(G) = 4$.*
(3) *For type G_2: $r(G) = 3$.*

We indicate the proof in the case of G_2. Let the notations be as above. Let P_1 be any positive system containing $P_\mathfrak{k}$. Let $\{\alpha, \beta\}$ be the simple roots of P_1. There are two possibilities: i) Only one of the simple roots is non compact ii) Both of them are non compact. In any case, if $\mathfrak{q}$ is the parabolic subalgebra associated to a simple root, then the unipotent radical $\mathfrak{u}$ of $\mathfrak{q}$ will contain at least 3 non compact roots. This proves that $r(G) = 3$ in this case.

The interested reader can derive vanishing theorems for other real simple non compact Lie groups, proceeding as above.

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