A GENERALIZATION OF THE ENRIGHT–VARADARAJAN MODULES

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For a semisimple Lie group admitting discrete series Enright and Varadarajan have constructed a class of modules, denoted $D_{P,\lambda}$ (cf. [3]). Their infinitesimal description based on the theory of Verma modules parallels that of finite dimensional irreducible modules. The introduction of the modules $D_{P,\lambda}$ in [3] was primarily to give an infinitesimal characterization of discrete series but we feel that [3] may well be a starting point for a fresh approach towards dealing with the problem of classification of irreducible representations of a general semisimple Lie algebra.

In order to give more momentum to such an approach we first construct modules which broadly generalize those in [3]. We briefly describe them now.

Let g_0 be any real semisimple Lie algebra, $g_0 = k_0 + p_0$ a Cartan decomposition and θ the associated Cartan involution. Let $g = k + p$ be the complexification. Let $U(g)$, $U(k)$ be the enveloping algebras of g, k respectively and let U^k be the centralizer of k in $U(g)$. For each θ stable parabolic subalgebra q of g we associate in this paper a class of irreducible k finite $U(g)$ modules having the following property: Like finite dimensional irreducible modules and like the Enright–Varadarajan modules $D_{P,\lambda}$, any member of this class comes with a special irreducible k-type occurring in it with multiplicity one, with an explicit description of the action of U^k on the corresponding isotypical k-type. We obtain these modules by extending the techniques in [3].

To see in what way these modules are related to the θ invariant parabolic subalgebra q we refer the reader to §2.

When our parabolic subalgebra q is minimal in g and when rank of g = rank of k, the class of $U(g)$ modules which we associate to this q coincides with the class of modules $D_{P,\lambda}$ of [3] (with a slight difference in parametrization). On the other hand when $q = g$ is the maximal parabolic subalgebra, the class we obtain is just the class of all finite dimensional irreducible representations of g. If k has trivial center, the trivial one dimensional $U(g)$ module is not equivalent to any of the modules $D_{P,\lambda}$ of [3]. This gap is bridged by the introduction of our class of $U(g)$ modules for every intermediate θ invariant parabolic subalgebra q between $q = g$ and $q = a$ θ invariant Borel subalgebra of g.

We have to point out that the knowledge of [3] is a necessary prerequisite to read this paper. If an argument or construction needed at some stage of this paper is parallel to that in [3] then instead of repeating them, we simply refer to [3].

§1. θ-stable parabolic subalgebras

As in the introduction, $g = k + p$ is the complexified Cartan decomposition arising from a real one $g_0 = k_0 + p_0$. Let θ be the Cartan involution. Let b be the complexification of a fixed Cartan subalgebra b_0 of k_0. Then the centralizer of b in g is a θ stable Cartan subalgebra h of g. We can write

$$h = b + a \tag{1.1}$$

where $a = p \cap h$. Let $a_0 = a \cap g_0$ and $h_0 = h \cap g_0$. Let Δ be the set of roots of (g, h). For α in Δ, denote by g^α the corresponding rootspace.

(1.2) LEMMA: *Let r_k be a Borel subalgebra of k containing b. Let q be a θ stable parabolic subalgebra of g containing h and assume that q contains r_k. Then q contains a θ stable Borel subalgebra r of g such that* (i) *$h \subseteq r$ and* (ii) *$r_k \subseteq r$.*

PROOF. Let u be the unipotent radical of q. Define a θ invariant element μ of $h^X (= \mathrm{Hom}_C(h, C))$ by $\mu(H) = \text{trace } (ad(H)|u)$. Let H'_μ in h be defined by $\lambda(H'_\mu) = (\lambda, \mu)$ for every λ in h^X. (Here and in the following the bilinear form is the nondegenerate one induced by the Killing form of g). Then

$$\theta(H'_\mu) = H'_\mu \text{ so } H'_\mu \in b. \tag{1.3}$$

Let

$$\Delta(q) = \{\alpha \in \Delta \,|\, \alpha(H'_\mu) \geq 0\}. \tag{1.4}$$

Then one can see that

(1.5) $$q = h + \sum_{\alpha \in \Delta(q)} g^{\alpha}.$$

Let C_k be the open Weyl chamber in ib_0 for (k, b) defined by the Borel subalgebra r_k. Since we assumed that $r_k \subseteq q$, it follows from 1.5 that

(1.6) $$H'_{\mu} \in \bar{C}_k = \text{the closure of } C_k.$$

Let α be in Δ. If α is identically zero on b, it would follow that b is not maximal abelian in k. Hence α is not identically zero on b. Let C'_k be the open subset of C_k got by deleting points of C_k where some α belonging to Δ vanishes. Then C'_k is the disjoint union

(1.7) $$C'_k = U_{j=1,\ldots,N} C'_{kj}$$

of its connected components and one has

(1.8) $$\bar{C}_k = U_{j=1,\ldots,N} \bar{C}'_{k,j}.$$

Choose an index M between 1 and N such that

(1.9) $$H'_{\mu} \in \bar{C}'_{k,M}.$$

Now choose an element X_j in $C'_{k,j}$ and consider the weight space decomposition of g with respect to $ad(X_j)$. We now define a Borel subalgebra r^j of g by

(1.10) r^j = the sum of the eigenspaces for $ad(X_j)$ with nonnegative eigenvalues.

If we define

(1.11) $$P^j = \{\alpha \in \Delta \mid \alpha(X_j) > 0\}$$

then clearly P^j is a positive system of roots in Δ and $r^j = h + \Sigma_{\alpha \in P^j} g^{\alpha}$. Since X_j belongs to k clearly both r^j and P^j are θ stable. 1.9 implies that for every α in P^M, $\alpha(H'_{\mu})$ is nonnegative. Hence from 1.4 and 1.5

(1.12) $$r^M \subseteq q.$$

Also since X_M belongs to C_k, (1.10) implies that

(1.13) $\qquad r_k$ is contained in r^M. $\qquad$ (q.e.d.)

(1.14) COROLLARY: *Let r_k be as in Lemma 1.2. Let r be a θ stable Borel subalgebra of g containing r_k. Then r equals one of the N Borel subalgebras r^j of* (1.10).

PROOF: Since r contains b, r contains a Cartan subalgebra of g containing b. h is the unique Cartan subalgebra of g containing b. Hence r contains h. In the proof of Lemma 1.2 take $q = r$. Then it is seen $r = r^M$.

(q.e.d.)

Rather than starting with a Borel subalgebra r_k of k containing b, we want to start with an arbitrary θ invariant parabolic subalgebra of g and recover the set up in Lemma 1.2. For this we prove the following lemma.

(1.15) LEMMA: *Let q be an arbitrary θ stable parabolic subalgebra of g. Then q contains a Borel subalgebra of k.*

PROOF: Let $Ad(g)$ be the adjoint group of g and Q the parabolic subgroup with Lie algebra q. Let G^u be the compact form of $Ad(g)$ with Lie algebra $k_0 + ip_0$. Note that G^u is θ-stable. It is well known that $G^u \cap Q$ is a compact form of a reductive Levi factor of Q (cf. [8, §1.2]). But $G^u \cap Q$ is θ stable since G^u and Q are θ stable. Thus, going to the Lie algebra level, q has a reductive Levi supplement which is θ stable. In this reductive Levi supplement we can surely find some θ stable Cartan subalgebra h' of g. Then, as in the proof of Lemma 1.2, we can find an element H'_μ in h' such that $\theta(H'_\mu) = H'_\mu$ and such that q is the sum of the nonnegative eigenspaces of $ad(H'_\mu)$. Since H'_μ lies in $h' \cap k$, clearly it follows that q contains a Borel subalgebra of k.

(q.e.d.)

(1.16) COROLLARY: *Let r by any θ stable Borel subalgebra of g. Then $r \cap k$ is a Borel subalgebra of k.*

§2. The objects r, r', P, P' and the choice of P'' associated with a θ stable parabolic subalgebra q

Now let q be a θ stable parabolic subalgebra of g. By (1.15) we can find a Borel subalgebra r_k of k contained in q. We fix a Cartan subalgebra b_0 of k_0 contained in r_k. Let a_0 be the centralizer of b_0 in p_0. Then $h_0 = b_0 + a_0$ is a θ stable Cartan subalgebra of g_0. Let $h = b + a$ be its complexification. Note that $h \subseteq q$. By (1.2), we can find a θ stable Borel subalgebra r of g such that $r_k \subset r$ and $r \subset q$. One has then $h \subset r$. There is a unique Borel subalgebra r' of g contained in q such that

(2.1) $\qquad r \cap r' = h + u$, where u is the unipotent radical of q.

Since $\theta(r')$ has the same property, we have $\theta(r') = r'$. Let $r'_k = r' \cap k$. Then by (1.16), r'_k is a Borel subalgebra of k. We observe that r'_k is the unique Borel subalgebra of k such that

(2.2) $\qquad r_k \cap r'_k = b + u_k$, where u_k is the unipotent radical of $q_k (= q \cap k)$.

We denote by W_k the Weyl group of (k, b) and by W_g the Weyl group of (g, h). W_k is naturally imbedded in W_g as follows: if s belongs to W_k then s normalizes b, hence also normalizes the centralizer of b in g which is precisely h. Thus s belongs to W_g.

We will now define two distinguished elements of the Weyl group W_k. Let t be the unique element of W_k such that $t(P_k) = -P_k$. Next we denote by τ the unique element of the Weyl group W_k such that $\tau(P_k) = P'_k$. The class of $U(g)$ modules associated to q will be parametrized by some subsets of h^X. We now prepare to describe these. Let Δ_k be the set of roots for (k, b). Whenever possible we will denote elements of Δ_k by φ while elements of Δ (= the roots of (g, h)) will be denoted by α. For a root φ in Δ_k, denote by X_φ a nonzero root vector in k of weight φ. For α in Δ, we denote by E_α a nonzero root vector in g of weight α. Let P and P' be the sets of positive roots in Δ defined respectively by r and r'. Next let P_k and P'_k be the sets of positive roots in Δ_k defined respectively by r_k and r'_k. Let δ and δ' denote half the sum of the roots in P and P' respectively and let δ_k and δ'_k denote half the sum of the roots in P_k and P'_k respectively.

Let P'' be a θ stable positive system of roots in Δ such that if r'' is the corresponding θ stable Borel subalgebra of g then

$$r'' \supseteq r'_k \text{ and} \tag{2.3}$$

$$P'' \supseteq P' \cap -P. \tag{2.4}$$

(2.5) REMARK: If one takes $P'' = P'$ then (2.3) and (2.4) are clearly satisfied. If q is a Borel subalgebra, then $P' = P$ and any P'' which satisfies (2.3) also satisfies (2.4). If $q = g$, then $P' = -P$; the only candidate which satisfies (2.3) and (2.4) is $P'' = P'$.

We can now describe the modules that we want to construct. As usual for α in P denote by H_α the element of $ib_0 + a_0$ such that $\lambda(H_\alpha) = 2(\lambda, \alpha)/(\alpha, \alpha)$ for every λ in h^X. Similarly for φ in P_k, denote by H^k_φ the element of ib_0 such that $\lambda(H^k_\varphi) = 2(\lambda, \varphi)/(\varphi, \varphi)$ for every λ in b^X. (Note: The Killing form of g induces a nondegenerate bilinear form on b which in turn induces one on b^X.)

Let $F(P'': q, r)$ be the set of all elements μ in h^X with the following properties:

(2.6) $\mu(H_\alpha)$ is a nonnegative integer for every α in P''.

(2.7) $\mu(H^k_\varphi)$ is nonzero for every φ in P_k and $\mu(H_\varphi)$ is nonzero for every α in $P \cap -P'$.

EXAMPLE: Suppose μ belonging to h^X is such that $\mu(H_\alpha)$ is a positive integer for every α in P''. Then one can show that μ belongs to $F(P'': q, r)$. The method of showing that $\mu(H^k_\varphi)$ is nonzero for every φ in P_k can be found in the proof of (3.6).

We now use some definitions and notations from [3, §§2, 5] (cf. also §§3, 5 here). Let U^k be the centralizer of k in $U(g)$. Let $\mu \in F(P'': q, r)$. Our aim is to construct a k-finite irreducible $U(g)$ module, denoted $D_{P'':q,r}(\mu)$ in which the irreducible k type with highest weight $-t(\tau\mu + \tau\delta - \tau\delta_k - \delta_k)$ (cf. 3.7) occurs with multiplicity one and such that on the corresponding isotypical $U(k)$ submodule, elements of U^k act by scalars given by the homomorphism $\chi_{P,-\mu-\delta}$ (cf. §5).

(2.8) REMARK: Fix q and r. For any compatible choice of P'' and for any element μ in $F(P'':q, r)$, we will show (cf. 3.6) that (i) $-\mu - \delta(H_\alpha)$ is a nonnegative integer for every α in $P \cap -P'$ and (ii) $\tau\mu + \tau\delta - \tau\delta_k - \delta_k(H^k_\varphi)$ is a nonnegative integer for every φ in P_k. Now define $\bar{F}(q, r)$ to consist of all μ in h^X satisfying (i) and (ii) above. In general $\bar{F}(q, r)$ properly contains $U_{P''}F(P'': q, r)$. Our constructions and proofs in §§3, 4, 5 go through perfectly well for any μ in $\bar{F}(q, r)$ and so we do have a k-finite irreducible $U(g)$ module in which the irreducible k type with highest weight $-t(\tau\mu + \tau\delta - \tau\delta_k - \delta_k)$ occurs with multiplicity one and such that on the corresponding isotypical $U(k)$ submodule elements of U^k act by scalars given by $\chi_{P,-\mu-\delta}$. We have restricted ourselves to the subsets $F(P'': q, r)$ rather than all of $\bar{F}(q, r)$ only because condition (ii) is the definition of $\bar{F}(q, r)$ is quite incomprehensible.

§3

Choose and fix an element μ in $F(P'': q, r)$ as in §2 (cf. (2.6) and (2.7)). For facts about Verma modules that we will be using we refer to [1, 2, 5, 6].

Let M be any $U(g)$ module. Let Q be a subset of Δ_k. An element v of M is said to be Q extreme if $X_\varphi \cdot v = 0$ for every φ in Q. For λ in b^X, v is called a weight vector of weight λ with respect to b if $H \cdot v = \lambda(H) \cdot v$ for all H in b. By $J(M)$ we denote the set of all λ in b^X for which there exists a nonzero weight vector of weight λ in M, which is P_k extreme where P_k is the positive system of roots in Δ_k defined in §2. For φ in Δ_k, M is said to be X_φ free if $X_\varphi \cdot v = 0$ implies $v = 0$. For a subalgebra s of g, M is said to be s-finite if every vector of M lies in a finite dimensional s submodule of M. For any η in π_k, let $m(\eta)$ denote the subalgebra of g spanned by the elements X_η, $X_{-\eta}$ and H_η^k. For the notion of $U(k)$ module of 'type P_k' we refer to [3, §2].

Let P_0 be a positive system of roots of Δ and let $\lambda \in h^X$. The Verma module $V_{g,P_0,\lambda}$ of $U(g)$ is defined as follows: It is the quotient of $U(g)$ by the left ideal generated by the elements $H - \lambda(H)$, $(H \in h)$ and $E_\alpha (\alpha \in P_0)$. The Verma modules of $U(k)$ are defined similarly. We will suppress g and write $V_{P_0,\lambda}$ for the Verma module $V_{g,P_0,\lambda}$.

We have the inclusions $h \subseteq r \subseteq q$ (cf. §2). Let π be the set of simple roots for P. The parabolic subalgebras of g containing r are in one to one correspondence with subsets of π. The subset of π corresponding to q is got as follows: Let σ in h^X be defined by $\sigma(H) = \text{trace}(ad\, H)|u$. Then

$$\pi(q) = \{\alpha \in \pi | (\sigma, \alpha) = 0\}$$

From standard facts about parabolic subalgebras (cf. [8, §1.2]) we know that elements of $P \cap -P'$ are of the form $\Sigma\, m_i \alpha_i$ where m_i are nonnegative integers and α_i are in $\pi(q)$. For α in Δ the element s_α of

W_g is the reflection corresponding to α. It is given by $s_\alpha(\lambda) = \lambda - 2(\lambda, \alpha)/(\alpha, \alpha) \cdot \alpha$. We now define a $U(g)$ module W_1 by

$$W_1 = V_{P,-\mu-\delta} \tag{3.3}$$

considered as a $U(k)$ module it has some nice properties.

(3.4) LEMMA: *W_1 considered as a module for $U(k)$ is a weight module with respect to b; i.e. W_1 is the sum of the weight spaces with respect to b. Denoting also $-\mu-\delta$ the restriction of $-\mu-\delta$ to b, all the weights are of the form $-\mu-\delta-\Sigma\, n_i\varphi_i$ where φ_i are elements of P and n_i are positive integers. Finally the weight spaces are finite dimensional and the weight space corresponding to $-\mu-\delta$ is one dimensional.*

PROOF: Since as a $U(g)$ module W_1 is the sum of weight spaces with respect to $h = b + a$, the first statement is clear. Since no root α in Δ is identically zero on b, we can pick up an element H in b such that for every α in P, $\alpha(H)$ is real and positive. As a $U(g)$ module, the weights of W_1 with respect to h are of the form $-\mu-\delta-\Sigma\, m_i\alpha_i$ ($\alpha_i \in P$, m_i nonnegative integers). By considering the action of H it is clear that weight spaces of W_1 with respect to b are finite dimensional and the weight space of b with weight $-\mu-\delta$ is one dimensional. Finally since P is θ stable the restriction to b of the weights with respect to h are of the form $-\mu-\delta-\Sigma\, n_i\varphi_i$ where φ_i are in P and n_i nonnegative integers.

(q.e.d.)

(3.5) COROLLARY: *The $U(k)$ submodule of W_1 generated by the unique weight vector in W_1 of weight $-\mu-\delta$ is isomorphic to the $U(k)$ Verma module $V_{k,P_k,-\mu-\delta}$. W_1 is $X_{-\varphi}$ free for every φ in P_k.*

PROOF: Let v_1 be the nonzero weight vector in W_1 of weight $-\mu-\delta \cdot v_1$ is killed by every element of $[r, r]$ hence in particular by every element of $[r_k, r_k]$. On the other hand let $\bar{r}$ be the unique Borel subalgebra of g such that $\bar{r} \cap r = h$ and let $n(\bar{r})$ be the unipotent radical of $\bar{r}$. If $\bar{r}_k = \bar{r} \cap k$, then $\bar{r}_k$ is the unique Borel subalgebra of k such that $\bar{r}_k \cap r_k = b$. Let $U(n(\bar{r}))$ and $U(n(\bar{r}_k))$ denote the corresponding enveloping algebras considered as subalgebras of $U(g)$. One knows that W_1 is $U(n(\bar{r}))$ free, [2]. Hence in particular it is $U(n(\bar{r}_k))$ free. The corollary now follows from [2, 7.1.8].

(q.e.d.)

There is an ascending chain of $U(k)$ Verma modules containing $V_{k,P_k,-\mu-\delta}$. This chain will give rise to a chain of $U(g)$ modules, which is fundamental in the work [3].

Recall the two distinguished elements t and τ of W_k from §2. The highest weight of the special irreducible representation of k which the $U(g)$ module $D_{P';q,r}(\mu)$ will contain is described in the corollary to the lemma below.

(3.6) LEMMA: (i) $-\mu-\delta(H_\alpha)$ *is a nonnegative integer for every* α *in* $P\cap -P'$ *and* (ii) $\tau\mu+\tau\delta-\tau\delta_k-\delta_k(H_\varphi^k)$ *is a nonnegative integer for every* φ *in* P_k.

PROOF: By (2.4), (2.7) and (2.8), one sees that $-\mu(H_\alpha)$ is a positive integer for every α in $P\cap -P'$. The elements of $P\cap -P'$ are nonnegative integral linear combinations of elements of $\pi(q)$. Since $\delta(H_\alpha)=1$ for every α in $\pi(q)$ it now follows that $-\mu-\delta(H_\alpha)$ is a nonnegative integer for every α in $P\cap -P'$.

To prove (ii) first suppose φ lies in $P'_k\cap P_k$. We will show that $\tau\mu-\delta_k(H_\varphi^k)$ and $\tau\delta-\tau\delta_k(H_\varphi^k)$ are both nonnegative integers. For this it is enough to show that $\tau\mu(H_\varphi^k)$ is a positive integer for every φ in P_k and that $\tau\delta(H_\varphi^k)$ is a positive integer for every φ in τP_k. By (2.6) there exists a finite dimensional representation of g having a weight vector v of weight μ with respect to the Cartan subalgebra h and such that v is annihilated by $[r'',r'']$ (cf. (2.3)). Since $r'_k\subseteq r''$, v is in particular annihilated by $[r'_k,r'_k]$. It is clear from this that $\mu(H_\varphi^k)$ is a nonnegative integer for every φ in P'_k. In view of (2.7), $\mu(H_\varphi^k)$ is then a positive integer for every φ in P'_k. Note that $\tau P'_k=P_k$. Hence $\tau\mu(H_\varphi^k)$ is a positive integer for every φ in P_k. It remains to show that $\tau\delta(H_\varphi^k)$ is a positive integer for every φ in τP_k. For this consider the representation ρ of g having a weight vector v of weight δ with respect to the Cartan subalgebra h and annihilated by $[r,r]$. Clearly then v is annihilated by $[r_k,r_k]$, hence $\delta(H_\varphi^k)$ is a nonnegative integer for every φ in P_k. To show that $\delta(H_\varphi^k)$ is nonzero we give the following reason: one can easily see that the stabilizer of v in g is exactly r. If $\delta(H_\varphi^k)$ is zero for some φ in P_k, then $X_{-\varphi}$ would stabilize v. But $X_{-\varphi}$ does not belong to r. Hence $\delta(H_\varphi^k)$ is a positive integer for every φ in P_k, so that $\tau\delta(H_k^k)$ is a positive integer for every φ in τP_k.

Now suppose φ lies in $P_k\cap -P'_k$. Let $r(q)$ be the maximal reductive subalgebra of q defined by $r(q)=h+\Sigma_{\alpha\in P\cap -P'}(g^\alpha+g^{-\alpha})$. By (ii) $-\mu-\delta(H_\alpha)$ is a nonnegative integer for every α in $P\cap -P'$. Hence, if $n_{r(q)}=\Sigma_{\alpha\in P\cap -P'}g^\alpha$, there exists a finite dimensional representation of $r(q)$ and a weight vector for h of weight $-\mu-\delta$ annihilated by all of

$n_{r(q)}$, hence in particular by $k \cap n_{r(q)}$. Observe that $P_k \cap -P'_k$ is precisely the set of roots in P_k, whose corresponding root spaces span $k \cap n_{r(q)}$. Thus there exists a finite dimensional representation of $b + \Sigma_{\varphi \in P_k \cap -P'_k}(C \cdot X_\varphi + C \cdot X_{-\varphi})$ with a weight vector for b of weight $-\mu - \delta$ annihilated by X_φ for every φ in $P_k \cap -P'_k$. Hence we conclude that $-\mu - \delta(H^k_\varphi)$ is a nonnegative integer for every φ in $P_k \cap -P'_k$. Since $-\tau(P_k \cap -P'_k) = P_k \cap -P'_k$, $\tau(\mu + \delta)(H^k_\varphi)$ is a nonnegative integer for every φ in $P_k \cap -P'_k$. On the other hand $\tau\delta_k = \delta'_k =$ half the sum of the roots in P'_k, while $\delta_k + \delta'_k(H^k_\varphi) = 0$ for every φ in $P_k \cap -P'_k$. Thus $\tau\mu + \tau\delta - \tau\delta_k - \delta_k(H^k_\varphi)$ is a nonnegative integer for every φ in $P_k \cap -P'_k$.

This completes the proof of (3.6). (q.e.d.)

(3.7) Corollary: *$-t(\tau\mu + \tau\delta - \tau\delta_k - \delta_k)(H^k_\varphi)$ is a nonnegative integer for every φ in P_k.*

Proof: Clear since $-tP_k = P_k$. (q.e.d.)

Let π_k be the set of simple roots of P_k. For φ in P_k, let s_φ be the reflection $s_\varphi(\lambda) = \lambda - \lambda(H_\varphi)\varphi$ of b^X. If φ lies in π_k, s_φ is called a simple reflection. For w in W_k, the length $N(w)$ of w is the smallest integer N such that w is a product of N simple reflexions. A reduced word for w is an expression of w as a product of $N(w)$ simple reflections. Choose any reduced word for the element τt of W_k. Following the notation in [5, §4.15], we write it as

$$\tau t = s_1 s_2 \ldots s_m \tag{3.8}$$

where $s_i = s_{\eta_i}$, $\eta_i = \varphi_{j_i}$, $\varphi_{j_i} \in \pi_k$. For λ in b^X and w in W_k write $w'(\lambda) = w(\lambda + \delta_k) - \delta_k$. Having chosen the element μ in $F(P'': q, r)$, we now define elements μ_i of b^X as follows:

$$\mu_{m+1} = -t(\tau\mu + \tau\delta - \tau\delta_k - \delta_k) \quad \text{and}$$

$$\mu_i = (s_i s_{i+1} \ldots s_m)' \mu_{m+1} \quad (i = 1, \ldots, m)$$

(3.9) Note that $\mu_1 = (\tau t)'\mu_{m+1} = -\mu - \delta$ and that μ_1 and μ_{m+1} are independent of the reduced expression (3.8). We now define the positive integers e_i by

$$e_i = \mu_{i+1} + \delta_k(H^k_{\eta_i}) \cdot (i = 1, \ldots, m). \tag{3.10}$$

With μ_i defined as above, the following inclusion relations between Verma modules are well known [2, 6]:

$$(3.11) \qquad V_{k,P_k,\mu_1} \subseteq V_{k,P_k,\mu_2} \subseteq \cdots \subseteq V_{k,P_k,\mu_{m+1}}.$$

Define elements $v_1, v_2, \ldots, v_{m+1}$ of $V_{k,P_k,\mu_{m+1}}$ as follows: v_{m+1} is the unique nonzero weight vector of $V_{k,P_k,\mu_{m+1}}$ of weight μ_{m+1}. For $i = 1, 2, \ldots, m$, $v_i = X_{-\eta_i}^{c_i} \cdot v_{i+1}$. Then one knows that v_i is of weight μ_i and that $V_{k,P_k,\mu_i} = U(k)v_i$. Associated to the reduced word (3.8) and μ in $F(P'' : q, r)$ is a fundamental chain of $U(g)$ modules: $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{m+1}$. It will turn out that W_1 and W_{m+1} are independent of the reduced expression (3.8). They are defined as follows: W_1 is defined to be $V_{P,-\mu-\delta}$ as in (3.3). Then W_{m+1} is given by the following lemma.

(3.12) LEMMA: *There exists a $U(g)$ module $W_{m+1} = U(g) \cdot v_{m+1}$ such that* (a) *W_1 is a $U(g)$ submodule of W_{m+1},* (b) *v_1 belongs to $U(k)v_{m+1}$,* (c) *v_{m+1} is a P_k extreme weight vector (with respect to* b) *of weight μ_{m+1},* (d) *W_{m+1} is $X_{-\varphi}$ free for all φ in P_k and* (e) *W_{m+1} is a sum of $U(k)$ submodules of type P_k.*

PROOF: Start with the inclusion of V_{k,P_k,μ_1} in W_1 given by Corollary 3.5 and the inclusion of V_{k,P_k,μ_1} in $V_{k,P_k,\mu_{m+1}}$ given by 3.11. By 3.5 we know that W_1 is $X_{-\varphi}$ free for every φ in P_k. Now [3, Lemma 4] gives us the module W_{m+1} with the properties required in the lemma. (One easily sees that the results of [3, §2] do not depend on the assumption there that rank of g = rank of k). (q.e.d.)

(3.13) REMARK: If V and $\bar{V}$ are Verma modules for, say, $U(k)$ then the space of $U(k)$ homomorphisms of V into $\bar{V}$ has dimension equal to zero or one. Thus the inclusion of V_{k,P_k,μ_1} into $V_{k,P_k,\mu_{m+1}}$ given by (3.11) is independent of the reduced expression (3.8) for τt. Hence also the $U(g)$ module W_{m+1} and the inclusion of W_1 in W_{m+1} with the properties listed in Lemma 3.12 can be chosen to be independent of the reduced expression (3.8).

Having defined W_1 and W_{m+1} as above, now for any given reduced word for τt such as (3.8), we define submodules $W_2, W_3, \ldots, W_m$ of W_{m+1} by

$$(3.14) \qquad W_i = U(g)v_i$$

where v_i are the elements of W_{m+1} defined after (3.11). We have

$W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{m+1}$ because v_i belongs to $U(k)v_{i+1}$, $(i = 1, \ldots, m)$. The properties of this chain of $U(g)$ modules are summarized below from the work of [3, §3]:

(3.15) $W_1 = V_{P,-\mu-\delta}$ and each W_i is the sum of its weight spaces with respect to b. Moreover as a $U(k)$ module W_i is the sum of $U(k)$ submodules of type P_k.

(3.16) Each W_i is a cyclic $U(g)$ module with a cyclic vector v_i, which is a P_k extreme weight vector of weight μ_i with respect to b, $i = 1, \ldots, m+1$.

(3.17) The P_k extreme vectors of weight μ_i in W_i are scalar multiples of v_i; for $i = 1, \ldots, m+1$, the vector v_i does not belong to W_{i-1}.

(3.18) Each W_i is $X_{-\varphi}$ free for every φ in P_k and W_{i+1}/W_i is $m(\eta_i)$ finite $(i = 1, \ldots, m)$.

(3.19) $$v_i = X_{-\eta_i}^{c_i} v_{i+1} \ (i = 1, \ldots, m).$$

(3.20) Let w be in W_k. Let $i = 1, \ldots, m$. Suppose $w'(\mu_{m+1})$ belongs to $J(W_i)$. Then $N(w)$ equals at least $m + 1 - i$.

We will not prove the properties (3.15) to (3.20) here since they are essentially proved in [3, Lemma 5]. Though (3.20) has the same form as [3, Lemma 5, vi] its proof is different in our case. It is important to first know the case $i = 1$ of (3.20) to carry over the inductive arguments of [3, §3] to our situation. To this end we prove the following lemma. Before that we make the following remark.

(3.21) Remark: Let H_q' be the element of h defined by $(H_q', H) =$ trace $(ad\, H|u)$, for every H belonging to h, where u is the unipotent radical of q. Since q and h are θ invariant $\theta(H_q') = H_q'$; hence H_q' belongs to b. One can easily prove the following: For every α in $P \cap -P'$, $\alpha(H_q')$ equals zero; for every α in $P \cap P'$, $\alpha(H_q')$ is a positive real number; and for every φ in $P_k \cap -P_k'$, $\varphi(H_q')$ equals zero while for every φ in $P_k \cap P_k'$, $\varphi(H_q')$ is a positive real number. (Observe that any φ in $P_k \cap -P_k'$ is the restriction to b of some α in $P \cap -P'$).

Now we come to the lemma which is basic to carry over the inductive arguments of [3, §3].

(3.22) LEMMA: *Let w be in W_k. Suppose $w'(\mu_{m+1})$ belongs to $J(W_1)$. Then $N(w)$ is greater than or equal to m.*

PROOF: Since $w'(\mu_{m+1})$ belongs to $J(W_1)$ it is in particular a weight of W_1 of for b. Hence by (3.4), $w'(\mu_{m+1})$ is of the form $\mu_1 - \Sigma\, n_i\alpha_i|b$, where n_i are nonnegative integers and α_i are in P. That is $w(\mu_{m+1} + \delta_k) - \delta_k = \mu_1 - \Sigma\, n_i\alpha_i|b = \tau t(\mu_{m+1} + \delta_k) - \delta_k - \Sigma\, n_i\alpha_i b$. Thus,

$$\tau t(\mu_{m+1} + \delta_k) - w(\mu_{m+1} + \delta_k) = \Sigma\, n_i\alpha_i|b.$$

Write $\mu'_{m+1} = -t\mu_{m+1}$. Hence

$$-\tau(\mu'_{m+1} + \delta_k) + wt(\mu'_{m+1} + \delta_k) = \sum n_i\alpha_i|b \tag{3.23}$$

where n_i are nonnegative integers and α_i are in P. The left side of the equality in (3.23) is the sum of $wt(\mu'_{m+1} + \delta_k) - (\mu'_{m+1} + \delta_k)$ and $(\mu'_{m+1} + \delta_k) - \tau(\mu'_{m+1} + \delta_k)$. We claim that (3.23) implies

$$P_k \cap -wt\, P_k \text{ is contained in } P_k \cap -\tau P_k. \tag{3.24}$$

To see this enumerate the elements of $P_k \cap -wt\, P_k$ in a sequence $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ such that ϵ_1 is a simple root of P_k and ϵ_{i+1} is a simple root of $s_{\epsilon_i} \ldots s_{\epsilon_1} P_k$ $(i = 1, \ldots, k-1)$. Then $wt = s_{\epsilon_k} \ldots s_{\epsilon_1}$ (cf. [5, 4.15.10] and [7, 8.9.13]). By induction on i one can show that $(\mu'_{m+1} + \delta_k) - s_{\epsilon_i} \ldots s_{\epsilon_1} (\mu'_{m+1} + \delta_k)$ can be written as $\Sigma_{j=1}^{i} d_{j,i}\epsilon_j$ where $d_{j,i}$ are positive integers. Thus $(\mu'_{m+1} + \delta_k) - wt(\mu'_{m+1} + \delta_k)$ can be written as $d_1\epsilon_1 + d_2\epsilon_2 + \cdots + d_k\epsilon_k$ where d_i are positive integers. Similarly $(\mu'_{m+1} + \delta_k) - \tau(\mu'_{m+1} + \delta_k)$ can be written as $d'_1\epsilon'_1 + d'_2\epsilon'_2 + \cdots + d'_h\epsilon'_h$ where d'_i are positive integers and $(\epsilon'_1, \ldots, \epsilon'_h)$ is an enumeration of $P_k \cap -\tau P_k$. With these observations we can write

$$\begin{aligned} -\tau(\mu'_{m+1} + \delta_k) &+ wt(\mu'_{m+1} + \delta_k) \\ &= (d'_1\epsilon'_1 + \cdots + d'_h\epsilon'_h) - (d_1\epsilon_1 + \cdots + d_k\epsilon_k) \end{aligned} \tag{3.25}$$

where $d'_1, \ldots, d'_h, d_1, \ldots, d_k$ are positive integers. Let H'_q be the element of h defined by $(H'_q, H) = \text{trace}(ad\, H|u)$, where u is the unipotent radical of q. Then H'_q belongs to b. We can apply remark (3.21) to (3.25) and conclude that $[-\tau(\mu'_{m+1} + \delta_k) + wt(\mu'_{m+1} + \delta_k)](H'_q)$ is a strictly negative real number unless (3.24) holds. But by looking at the right hand side of (3.23) and applying remark (3.21), we see that $[-\tau(\mu'_{m+1} + \delta_k) + wt(\mu'_{m+1} + \delta_k)](H'_q)$ is a nonnegative real number.

Thus we have proved the validity of (3.24). Now (3.24) implies that $N(wt)$ is less than or equal to $N(\tau)$. But note that $N(wt) = N(t) - N(w)$, while $N(\tau) = N(t) - N(\tau t) = N(t) - m$. Hence $N(w)$ is greater than or equal to m.

(q.e.d.)

(3.22) enables us to carry over the inductive arguments in [3, §3] without any further change and obtain the properties (3.15) to (3.20).

§4. The k-finite quotient $U(g)$ module of W_{m+1}

The difference between the special situation in [3] and our more general situation becomes more apparent in this section which parallels [3, §4].

Start with an arbitrary reduced word (3.8) for τt and let $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{m+1}$ be a fundamental chain of $U(g)$ modules satisfying (3.15) through (3.20). Recall $W_1 = V_{P,-\mu-\delta}$. Recall the subset $\pi(q) \subseteq \pi$ corresponding to the parabolic subalgebra q. For α in π and λ in h^X define $s_\alpha^X(\lambda) = s_\alpha(\lambda + \delta) - \delta$. By Lemma 3.6, $-\mu - \delta(H_\alpha)$ is a nonnegative integer for every α in $P \cap -P'$, hence in particular for every α in $\pi(q)$. Thus one has the inclusion of the Verma modules $V_{P,s_\alpha^X(-\mu-\delta)} \subseteq V_{P,-\mu-\delta}$ for every α in $\pi(q)$. We now define a $U(g)$ submodule

$$(4.1) \qquad W_0 = \sum_{\alpha \in \pi(q)} V_{P,s_\alpha^X(-\mu-\delta)} \text{ of } W_1.$$

As is well known the Verma modules have unique proper maximal submodules. Let I be the proper maximal $U(g)$ submodule of $V_{P,-\mu-\delta}$. Then each $V_{P,s_\alpha^X(-\mu-\delta)}$ $(\alpha \in \pi(q))$ is contained in I. Hence

$$(4.2) \qquad v_1 \text{ does not belong to } W_0.$$

Now fix some i, $(i = 1, \ldots, m)$. Define a $U(g)$ submodule (relative to some reduced word (3.8) for τt) $\bar{W}_i$ of W_{m+1} as follows: Let $W_{i,0}$ be the $U(g)$ submodule of all vectors in W_{m+1} that are $m(\eta_i)$ finite mod W_{i-1}; once $W_{i,0}, \ldots, W_{i,p-1}$ are defined, $W_{i,p}$ is the $U(g)$ submodule of all vectors in W_{m+1} that are $m(\eta_{i+p})$ finite mod $W_{i,p-1}$, $p = 1, 2, \ldots, m - i$. We have $W_{i,0} \subseteq \cdots \subseteq W_{i,m-i}$. We then define $\bar{W}_i = W_{i,m-i}$. Define

$$(4.3) \qquad \bar{W} = W_m + \bar{W}_1 + \bar{W}_2 + \cdots + \bar{W}_m.$$

Thus for each reduced expression (3.8) for τt, we have defined a $U(g)$ submodule $\bar{W}$ of W_{m+1}.

(4.4) PROPOSITION: *For any reduced word (3.8) for τt, define the $U(g)$ submodule $\bar{W}$ of W_{m+1} as above. Then v_{m+1} does not belong to $\bar{W}$. If $\lambda \in b^X$ is such that W_{m+1} has a nonzero P_k extreme weight vector (with respect to b) of weight λ which is nonzero* mod $\bar{W}$, *then $(\tau t)'\lambda$ is a P_k extreme weight of W_1/W_0.*

PROOF: We refer to the proof of [3, Lemma 9].

Since we do not have a full chain of $U(g)$ modules corresponding to a reduced word for t as in [3] but only a shorter chain corresponding to a reduced word for τt, we have to work more to obtain a k-finite quotient $U(g)$ module of W_{m+1}. We now define

(4.5) $W_X = \Sigma\, \bar{W}$, the summation being over all reduced expressions (3.8) for τt.

(4.6) LEMMA: *v_{m+1} does not belong to W_X.* Let *$\lambda \in b^X$ be such that there is a P_k extreme vector in W_{m+1} of weight λ which is nonzero* mod *W_x. Then $(\tau t)'\lambda(H^k_\varphi)$ is a nonnegative integer for every φ in $P_k \cap -P'_k$.*

PROOF: v_{m+1} is a P_k extreme weight vector in W_{m+1} of weight μ_{m+1}. From (3.7) and the definition of μ_{m+1}, we know that $\mu_{m+1}(H^k_\varphi)$ is a nonnegative integer for every φ in P_k. Now suppose v_{m+1} belongs to W_X. Since $W_X = \Sigma\, \bar{W}$, W_X is a quotient of the abstract direct sum $\oplus\, \bar{W}$, the summation being over all reduced words (3.8) for τt. We can then apply [3, Lemma 7] and conclude that for some reduced word (3.8) for τt, the corresponding $\bar{W}$ has a nonzero P_k extreme vector of weight μ_{m+1}. This vector has to be a nonzero scalar multiple of v_{m+1} in view of (3.17). Hence v_{m+1} belongs to *that* $\bar{W}$. But this contradicts (4.4). This proves the first assertion in (4.6).

Next let λ be as in the lemma. Let c be the reductive component of q defined by $c = h + \Sigma_{\alpha \in P \cap -P'}(g^\alpha + g^{-\alpha})$. *We claim that W_1/W_0 is c-finite.* For this it is enough to show that the image $\bar{v}_1$ in W_1/W_0 of v_1 is c-finite. For any α in $\pi(q)$ the submodule $V_{g,P,s^1_\alpha(\mu_1)}$ of W_1 coincides with $U(g)X_{-\alpha}^{\mu_1(H_\alpha)+1} \cdot v_1$ (cf. [2, 7.1.15]). Thus we have $W_0 = \Sigma_{\alpha \in \pi(q)} U(g)X_{-\alpha}^{\mu_1(H_\alpha)+1} \cdot v_1$. Hence the annihilator in $U(g)$ of $\bar{v}_1$ contains $U(g)X_{-\alpha}^{\mu_1(H_\alpha)+1}$ for every α in $\pi(q)$. This suffices in view of [2, 7.2.5] to conclude that $\bar{v}_1$ is c-finite. Thus W_1/W_0 is c-finite.

Let $c_k = c \cap k$. Then in particular W_1/W_0 is c_k-finite. But note that $c_k = b + \Sigma_{\varphi \in P_k \cap -P'_k}(C \cdot X_\varphi + C \cdot X_{-\varphi})$.

Now choose some reduced word (3.8) for τt and relative to it define $\bar{W}$ as in (4.3). Note that $\bar{W} \subseteq W_X$. For λ as in the lemma, choose a P_k extreme weight vector v in W_{m+1} which is nonzero mod W_X and is of weight λ. Then v is in particular nonzero mod $\bar{W}$. Hence from (4.4), $(\tau t)'\lambda$ is a P_k extreme weight of W_1/W_0. Since W_1/W_0 is c_k-finite, it now follows that $(\tau t)'\lambda(H_\varphi^k)$ is a nonnegative integer for every φ in $P_k \cap -P'_k$.

(q.e.d.)

For our proof of the k-finiteness of W_{m+1}/W_k, we need one more lemma.

(4.7) Lemma: *Let η be in b^X. Suppose $\eta(H_\varphi^k)$ is nonnegative for every φ in P_k. Let s be in W_k. Suppose $(\tau ts)'\eta(H_\varphi^k)$ is nonnegative for every φ in $P_k \cap -P'_k$. Then $N(\tau t) = N(\tau ts) + N(s^{-1})$.*

Proof: $(\tau ts)'\eta = \tau ts(\eta + \delta_k) - \delta_k$. Since $\eta(H_\varphi^k)$ is nonnegative for every φ in P_k, $\tau ts(\eta + \delta_k)(H_\varphi^k)$ is negative for every φ in $-\tau ts\, P_k$. Also $-\delta_k(H_\varphi^k)$ is negative for every φ in P_k. Hence $(\tau ts)'\eta(H_\varphi^k)$ is negative for every φ in $(-\tau ts\, P_k) \cap P_k$. Hence the assumption implies

$$P_k \cap -\tau ts\, P_k \subseteq \text{complement of } P_k \cap -P'_k \text{ in } P_k. \tag{4.8}$$

Note that $tP_k = -P_k$ and $\tau P_k = P'_k$. So, $-P'_k = \tau t P_k$. So, the complement of $P_k \cap -P'_k$ in P_k is $P_k \cap -\tau t P_k$. Hence from (4.8) we have

$$P_k \cap (-\tau ts P_k) \subseteq P_k \cap (-\tau t P_k). \tag{4.9}$$

Let $(\epsilon_1, \epsilon_2, \ldots, \epsilon_j, \epsilon_{j+1}, \ldots, \epsilon_m)$ be an enumeration of the elements of $(-\tau t P_k) \cap P_k$ such that ϵ_1 is a simple root of P_k, ϵ_2 is a simple root of $s_{\epsilon_1}P_k, \ldots, \epsilon_{i-1}$ is a simple root of $s_{\epsilon_i}s_{\epsilon_{i-1}} \ldots s_{\epsilon_1}P_k$ $(i = 1, \ldots, m-1)$. Because of (4.9) we can further assume $(\epsilon_1, \ldots, \epsilon_j)$ is an enumeration of $(-\tau ts P_k) \cap P_k$. Let

$$\varphi'_i = s_{\epsilon_1} \ldots s_{\epsilon_{i-1}}(\epsilon_i) \quad (i = 1, \ldots, m) \quad (\varphi'_1 = \epsilon_1).$$

Then φ'_i belongs to π_k. One can show that $\tau t = s_{\epsilon_m} \ldots s_{\epsilon_1}$ and a reduced word for τt is

$$\tau t = s_{\varphi'_1} s_{\varphi'_2} \ldots s_{\varphi'_m} \tag{4.10}$$

(cf. [5, 4.15.10] and [7, 8.9.13]). Similarly $\tau ts = s_{\epsilon_j} \dots s_{\epsilon_1}$ and a reduced word for τts is

$$\tau ts = s_{\varphi_1'} \dots s_{\varphi_j'}. \tag{4.11}$$

Note that $N(\tau t) = m$ and $N(\tau ts) = j$. Now from (4.10) and (4.11) it is clear that $s^{-1} = s_{\varphi_{j+1}'} \dots s_{\varphi_m'}$ is a reduced word for s^{-1}. These observations substantially prove the lemma.

(q.e.d.)

(4.12) REMARK: With the data assumed in Lemma 4.7 we have actually proved more than what is asserted in (4.7): There exists a reduced word $\tau t = s_{\varphi_1'} \dots s_{\varphi_j'} s_{\varphi_{j+1}'} \dots s_{\varphi_m'}$ for τt such that $s^{-1} = s_{\varphi_{j+1}'} \dots s_{\varphi_m'}$.

The following proposition gives the k-finite $U(g)$ module quotient of W_{m+1}.

(4.13) PROPOSITION: *The $U(g)$ module W_{m+1}/W_X is k-finite.*

PROOF: Let $\bar{v}_{m+1}$ be the image of v_{m+1} in W_{m+1}/W_X. Since $U(g)\bar{v}_{m+1} = W_{m+1}/W_X$, it suffices to prove that $U(k) \cdot \bar{v}_{m+1}$ has finite dimension over C. For this again, by well known facts [2, 7.2.5] it suffices to prove that the annihilator of $\bar{v}_{m+1}$ in $U(k)$ contains $X_{-\varphi}^{e(\varphi)}$ for every φ in π_k, where $e(\varphi) = \mu_{m+1}(H_\varphi^k) + 1$ (observe that in view of (3.7), $\mu_{m+1}(H_\varphi^k)$ is a nonnegative integer for every φ in π_k). Thus it suffices to show that for every φ in π_k,

$$X_{-\varphi}^{e(\varphi)} \cdot v_{m+1} \text{ belongs to } W_X. \tag{4.14}$$

Suppose (4.14) *is not true.* Choose a φ in π_k, such that $X_{-\varphi}^{e(\varphi)} v_{m+1}$ does not belong to W_X. Then $X_{-\varphi}^{e(\varphi)} v_{m+1}$ is a P_k extreme vector of weight $s_\varphi'(\mu_{m+1})$ in W_{m+1} which is nonzero mod W_X. Hence by (4.6), $(\tau ts_\varphi)'\mu_{m+1}(H_{\varphi'}^k)$ is a nonnegative integer for every φ' in $P_k \cap -P_k'$. We can now apply (4.7) and (4.12) and conclude that there exists a reduced word

$$\tau t = s_{\varphi_1'} s_{\varphi_2'} \dots s_{\varphi_{m-1}'} s_{\varphi_m'} \quad (\varphi_i' \in \pi_k) \tag{4.15}$$

for τt such that

$$\varphi_m' = \varphi. \tag{4.16}$$

Take the reduced word (4.15) for τt in (3.8) and consider the corresponding modules W_m and $\bar{W}$. By definition $W_m \subseteq \bar{W}$. But in the fundamental chain $W_1 \subseteq \cdots \subseteq W_m \subseteq W_{m+1}$ associated to the reduced word (4.15) for τt, the module W_m is simply $U(g) \cdot X_{-\varphi}^{e(\varphi)} v_{m+1}$. This is clear from the definitions (cf. (3.14) and the definition of v_i after (3.11)) and (4.16). Thus it follows that $X_{-\varphi}^{e(\varphi)} v_{m+1} \in \bar{W} \subseteq W_X$. But this is a contradiction to the hypothesis. Thus (4.14) is true and proved and with that also the k-finiteness of W_{m+1}/W_X.

(q.e.d.)

§5

Let b be a Cartan subalgebra of k and h its centralizer in g, so that h is a θ stable Cartan subalgebra of g. Let P be a system of positive roots for (g, h) such that $\theta(P) = P$. Let

$$n^+ = \sum_{\alpha \in P} g^\alpha$$

and

$$n^- = \sum_{\alpha \in P} g^{-\alpha}.$$

The following fact is standard if $b = h$, but it remains true in our general case.

(5.1) LEMMA: *Let U^b be the centralizer of b in $U(g)$. If the set P of positive roots satisfies $\theta P = P$, we have a unique homomorphism*

(5.2) $$\beta_P : U^b \to U(h)$$

such that for any y in U^b

(5.3) $$y = \beta_P(y)(\text{mod } U(g)n^+).$$

PROOF: We have

(5.4) $$U(g) = U(n^- + h) \oplus U(g)n^+$$

and this decomposition is stable under $ad\,H$ for every H in h, i.e. $ad\,H(U(n^- + h)) \subseteq U(n^- + h)$ and $ad\,H(U(g)n^+) \subseteq U(g)n^+$. For y in U^b, let $y = y_0 + y_1$ be its decomposition with respect to (5.4). Define

$\beta_P(y) = y_0$. *We claim* $\beta_P(y)$ *belongs to the subalgebra* $U(h)$ *of* $U(n^- + h)$. Since y is in U^b, y_0 and y_1 are also in U^b. Let $S(n^- + h)$ and $S(h)$ denote the symmetric algebras and λ the symmetrizer map of $S(n^- + h)$ onto $U(n^- + h)$. Then for H in b, $\lambda^{-1}(y_0)$ is annihilated by $ad\,H$ (extended as a derivation to $S(n^- + h)$). It is enough to show that $\lambda^{-1}(y_0)$ belongs to $S(h)$. Using (1.14), one can show that there exists an element X_P in b such that $\alpha(X_P)$ is a nonzero real number for every α in Δ (= the roots of (g, h)) and such that P consists of precisely those α in Δ such that $\alpha(X_P)$ is positive. It is then clear that in $S(n^- + h)$, the null space for $ad\,X_P$ is just $S(h)$. Since $ad\,X(\lambda^{-1}(y_0)) = 0$ for every X in b, in particular $ad\,X_P(\lambda^{-1}(y_0)) = 0$. Hence $\lambda^{-1}(y_0)$ belongs to $S(h)$, so that $\beta_P(y)$ belongs to $U(h)$.

Now suppose y and y' are in U^b. Let $y = y_0 + y_1$ and $y' = y_0' + y_1'$ be their decomposition with respect to (5.4), so that $\beta_P(y) = y_0$ and $\beta_P(y') = y_0'$. Then $yy' = y_0y_0' + y_0y_1' + y_1y_0' + y_1y_1'$. Clearly y_0y_0' belongs to $U(h)$ and $y_0y_1' + y_1y_1'$ belongs to $U(g)n^+$. Also $y_1y_0' \in U(g)n^+ \cdot U(h) \subseteq U(g)U(h)n^+$. Thus y_0y_0' is the component of yy' in $U(n^- + h)$ with respect to (5.4). We already know that this component is in $U(h)$. Thus β_P is a homomorphism of algebras. (q.e.d.)

The centralizer U^k of k in $U(g)$ is contained in U^b. As usual interpret elements of $S(h)$ as polynomials on h^X. For any φ in h^X, define a homomorphism $\chi_{P,\varphi}$ of U^k into C as follows:

$$\chi_{P,\varphi}(y) = \beta_P(y)(\varphi) \quad (y \in U^k). \tag{5.5}$$

The main results of the previous sections can now be formulated.

Let b_0 be a Cartan subalgebra of k_0 and b its complexification. Let q be a θ stable parabolic subalgebra of g containing b. The centralizer h of b in g is a Cartan subalgebra of g and q contains h. Let r be a θ stable Borel subalgebra of g contained in q (cf. (1.15) and (1.2)). Let P be the set of positive roots for (g, h) corresponding to r. Define the θ stable Borel subalgebra $r' \subseteq q$ by (2.1). Choose a θ stable positive system P'' of roots of (g, h) having properties (2.3) and (2.4). Denote by $F(P'': q, r)$ the set of all elements μ in h^X having properties (2.6) and (2.7). Now choose a μ in $F(P'': q, r)$ and recall the objects associated to it in §§3, 4.

We can now state

(5.6) THEOREM: *Let* q *be a* θ *stable parabolic subalgebra. Let* $\mu \in F(P'': q, r)$. *Let* $W_{P'':q,r} = W_{m+1}/W_X$ (cf. (3.12) and (4.5)). Then $W_{P'':q,r}(\mu)$ *is a* k *finite* $U(g)$ *module having the following properties*:

(i) $W_{P'':q,r}(\mu) = U(g)\bar{v}_{m+1}$, *where* $\bar{v}_{m+1}$ *is the image of the vector* v_{m+1} *of* W_{m+1}. *The irreducible finite dimensional representation of* k *with highest weight* $-t(\tau\mu + \tau\delta - \tau\delta_k - \delta_k)$ *occurs with multiplicity one in* $W_{P'':q,r}(\mu)$. *The corresponding isotypical* $U(k)$ *submodule of* $W_{P'':q,r}$ *is* $U(k)\bar{v}_{m+1}$; *on this space elements of* U^k *act by scalars given by the homomorphism* $\chi_{P,-\mu-\delta}$.

(ii) *If* τ_λ *is an irreducible finite dimensional representation of* k *with highest weight* λ *with respect to* P_k, *then the multiplicity of* τ_λ *in* $W_{P'':q,r}(\mu)$ *is finite; it is zero if* λ *is not of the form* $(t\tau)'(-\mu - \delta - \Sigma_{\varphi\in P} m_\varphi \varphi)|b$ *where* m_φ *are nonnegative integers.*

PROOF: By (4.13), we know that $W_{P'':q,r}(\mu)$ is nonzero and k-finite. By (4.6) the vector v_{m+1} of W_{m+1} does not belong to W_χ. The image of v_{m+1} in $W_{P'':q,r}(\mu)$ is P_k extreme of weight $(t\tau)'(-\mu-\delta) = -t(\tau\mu + \tau\delta - \tau\delta_k - \delta_k)$ (which is dominant by (3.7)) and this image generates an irreducible k-module with highest weight $-t(\tau\mu + \tau\delta + \tau\delta_k - \delta_k)$ with respect to P_k.

Based on the preceding sections one can complete the proof of the theorem in the same way as [3, Theorem 1].

It is easy to conclude from (5.6) that $W_{P'':q,r}(\mu)$ has a unique proper maximal $U(g)$ submodule and hence $W_{P'':q,r}(\mu)$ has a unique nonzero quotient $U(g)$ module which is irreducible. We denote this $U(g)$ module by $D_{P'':q,r}(\mu)$. The following theorem is now immediate from (5.6).

(5.7) THEOREM: *Let* $\mu \in F(P'': q, r)$. *Up to equivalence there exists a unique* k*-finite irreducible* $U(g)$ *module* $D_{P'':q,r}(\mu)$ *having the following property: The finite dimensional irreducible* $U(k)$ *module with highest weight* $-t(\tau\mu + \tau\delta - \tau\delta_k - \delta_k)$ *(with respect to* P_k*) occurs with multiplicity one in* $D_{P'':q,r}(\mu)$ *and the action of* U^k *on the corresponding isotypical* $U(k)$ *submodule is given by the homomorphism* $\chi_{P,-\mu-\delta}$.

The uniqueness follows from the well known theorem of Harish Chandra [4]: An irreducible k-finite $U(g)$ module M is completely determined by a nonzero isotypical $U(k)$ submodule of M and the action of U^k on it.

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