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Dirac operator and the discrete series

By R. PARTHASARATHY

Introduction

Let G be a noncompact semisimple Lie group with a finite dimensional faithful representation and K a maximal compact subgroup of G. Throughout this paper we assume that rank of K = rank of G. When G/K is hermitian symmetric, M. S. Narasimhan and K. Okamoto (see [9]) constructed most of the discrete series for G on spaces of square integrable harmonic forms of type $(0, q)$ (for suitable q) with coefficients in holomorphic vector bundles on G/K arising from finite dimensional irreducible unitary representations of K.

In general (i.e. when G/K is not assumed to be hermitian symmetric) we give in this paper an analogous procedure for constructing the discrete series for G, which we will now briefly describe. We assume (by going to a finite covering of G, if necessary) that if $\mathfrak{p}$ is the tangent space at $\{K\} \in G/K$, then the isotropy homomorphism $\alpha: K \to SO(\mathfrak{p})$ lifts to a homomorphism $\tilde{\alpha}: K \to \mathrm{Spin}(\mathfrak{p})$. (Here $SO(\mathfrak{p})$ is the rotation group of $\mathfrak{p}$ for a K invariant metric on $\mathfrak{p}$ and $\mathrm{Spin}(\mathfrak{p})$ the connected double covering of $SO(\mathfrak{p})$.) Then G/K acquires a G invariant spin structure (see § 1). Now let $\sigma: \mathrm{Spin}(\mathfrak{p}) \to \mathrm{Aut}(L)$ be the spin representation and $\sigma^{\pm}: \mathrm{Spin}(\mathfrak{p}) \to \mathrm{Aut}(L^{\pm})$ the two half spin representations of $\mathrm{Spin}(\mathfrak{p})$. Let $\chi: K \to \mathrm{Aut}(L)$ and $\chi^{\pm}: K \to \mathrm{Aut}(L^{\pm})$ be the composite homomorphisms $\sigma \circ \tilde{\alpha}$ and $\sigma^{\pm} \circ \tilde{\alpha}$ respectively. Now let $\tau: K \to \mathrm{Aut}(V)$ be a finite dimensional irreducible unitary representation of K. Let $C^{\pm}(E_V)$ be the space of C^{∞} sections of the bundle $E_{L^{\pm} \otimes V}$ on G/K induced by $\chi^{\pm} \otimes \tau$. Let $D: C^{\pm}(E_V) \to C^{\mp}(E_V)$ denote the Dirac operator arising from this spin structure on G/K (see 1.12) and let $H_2^{\pm}(E_V)$ be the space of square integrable sections annihilated by the Dirac operator D on $C^{\pm}(E_V)$. Our main result is stated in Theorem 3, § 8 which realizes most of the discrete series for G on $H_2^{\pm}(E_V)$ for suitable τ.

In the construction of the discrete series in [9] a key role is played by a formula (Theorem 4.1, in [11]) due to K. Okamoto and H. Ozeki, for the usual Laplacian operator $\square = (d'' + \delta'')^2$ on the spaces of vector valued C^{∞} forms of type $(0, q)$. This formula asserts that $2\,\square$ differs by a certain scalar multiplication from the action of the Casimir of G. Our main task is to provide such a formula (see Proposition 3.1) for D^2, the square of the Dirac operator, on $C^{\pm}(E_V)$. The proof of this formula utilizes the fact that the Casimir of K

acts as the same scalar on all the irreducible K-components of the representation χ. (See Lemma 2.2.)

The construction of the discrete class is achieved by first considering (Theorem 1, § 7) the difference Trace π^+ − Trace π^-, where Trace $\pi^\pm$ denotes the character of the representation $\pi^\pm$ of G on the spaces $H_2^\pm(E_V)$ and then proving (Theorem 2, § 8) that one of the two spaces $H_2^\pm(E_V)$ vanishes (only to be able to prove the vanishing part, we had to put the restriction 'most'). Our condition is less restrictive than that of [9].

W. Schmid has constructed (see [12]) most of the discrete series for G on "L^2-cohomology" groups of holomorphic line bundles on G/T, where T is a compact Cartan subgroup of G. R. Hotta has recently constructed most of the discrete series for G on certain eigenspaces of the Casimir of G, in the space of L^2-sections of some vector bundles on G/K (see [6]).

I am indebted to Professor M. S. Narasimhan for suggesting the use of the Dirac operator in realizing the discrete series. Further I am grateful to him for several profitable discussions. I thank the referee who offered simplifications of my proofs of Lemma 2.2 and Proposition 3.1, and who helped in getting the paper in the present form.

1. Spin structure on G/K and the Dirac operator

Let G be a connected noncompact semisimple Lie group with a finite dimensional faithful representation. Let $\mathfrak{g}$ be the Lie algebra of left invariant vector fields on G. Let $\mathfrak{k}$ be a maximal compactly imbedded subalgebra of $\mathfrak{g}$ (i.e. the analytic subgroup of Int($\mathfrak{g}$) corresponding to the subalgebra $\mathrm{ad}_{(\mathfrak{g})}\mathfrak{k}$ is a maximal compact subgroup of Int($\mathfrak{g}$)). Let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}$. Let $\mathfrak{g}^C$ be the complexification of $\mathfrak{g}$. We put

$$\mathfrak{p} = \{Y \in \mathfrak{g} \mid B(X, Y) = 0 \text{ for all } X \in \mathfrak{k}\}$$

where B denotes the Killing form of $\mathfrak{g}^C$. Then we have

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}\ ,\quad \mathfrak{k} \cap \mathfrak{p} = 0\ ,\quad [\mathfrak{p}, \mathfrak{p}] \subseteqq \mathfrak{k}\ ,\quad [\mathfrak{k}, \mathfrak{p}] \subseteqq \mathfrak{p}\ .$$

For any subset $\mathfrak{m}$ of $\mathfrak{g}^C$ we denote by $\mathfrak{m}^C$ the complex subspace of $\mathfrak{g}^C$ spanned by $\mathfrak{m}$.

We assume throughout rank of $\mathfrak{k}$ = rank of $\mathfrak{g}$. The restriction of B to $\mathfrak{p}$ is a positive definite real bilinear form. Let $SO(\mathfrak{p})$ be the rotation group of $\mathfrak{p}$ under this positive definite bilinear form. Under the adjoint action of K on $\mathfrak{g}$, $\mathfrak{p}$ is stable and we get a homomorphism

$$\alpha\colon K \longrightarrow SO(\mathfrak{p})\ .$$

Thus, we get a G-invariant Riemannian metric on the homogeneous space

G/K, which on the tangent space at $\{K\} \in G/K$ is $B|\mathfrak{p}$, when $\mathfrak{p}$ is identified with the tangent space at $\{K\} \in G/K$, in the usual way. We will now give a spin structure on the Riemannian manifold G/K.

By definition a 'spin structure' on an oriented Riemannian manifold M of dimension n is a principal $\mathrm{Spin}(n)$-bundle $\tilde{F}$ on M such that the principal $SO(n)$-bundle $\tilde{F} \times_{\mathrm{Spin}(n)} SO(n)$ is $SO(n)$ equivalent to the principal $SO(n)$-bundle F of oriented orthogonal frames of M. In our case F is the $SO(\mathfrak{p})$-bundle

$$G \times_K SO(\mathfrak{p})$$

induced from the principal K-bundle G, by the homomorphism $\alpha\colon K \to SO(\mathfrak{p})$. (We recall that $G \times_K SO(\mathfrak{p})$ is defined as the quotient of $G \times SO(\mathfrak{p})$ by the equivalence relation $(g, a) \sim (gk, k^{-1}a)$ for any $k \in K$, $g \in G$ and $a \in SO(\mathfrak{p})$. The equivalence class of (g, a) will be denoted by $\{g, a\}$.) For the spin structure on G/K we proceed as follows: First, we can assume by replacing G if necessary by a suitable covering that there exists a homomorphism

$$\tilde{\alpha}\colon K \longrightarrow \mathrm{Spin}(\mathfrak{p})$$

making the following diagram commutative:

$$\begin{array}{ccc} & & \mathrm{Spin}(\mathfrak{p}) \\ & \overset{\tilde{\alpha}}{\nearrow} & \downarrow \psi \\ K & \xrightarrow{\ \alpha\ } & SO(\mathfrak{p}) \end{array}$$

i.e. such that $\alpha = \psi \circ \tilde{\alpha}$. Here, $\psi\colon \mathrm{Spin}(\mathfrak{p}) \to SO(\mathfrak{p})$ is the connected two fold covering of $SO(\mathfrak{p})$. (Note that $\tilde{\alpha}$ is unique.) We can now take for $\tilde{F}$ the principal $\mathrm{Spin}(\mathfrak{p})$ bundle $G \times_K \mathrm{Spin}(\mathfrak{p})$, induced from the principal K-bundle G, via the homomorphism $\tilde{\alpha}\colon K \to \mathrm{Spin}(\mathfrak{p})$. This spin structure is G invariant in the following sense: G acts on the left on $G \times_K \mathrm{Spin}(\mathfrak{p})$ and also on $G \times_K SO(\mathfrak{p})$ and the canonical map

$$G \times_K \mathrm{Spin}(\mathfrak{p}) \longrightarrow G \times_K SO(\mathfrak{p})$$

defined by

$$\{g, a\} \longmapsto \{g, \psi(a)\}$$

commutes with the action of G.

Let $\sigma\colon \mathrm{Spin}(\mathfrak{p}) \to \mathrm{Aut}(L)$ be the (complex) spin representation of $\mathrm{Spin}(\mathfrak{p})$. Since we assumed rank of $\mathfrak{k}$ = rank of $\mathfrak{g}$, the dimension of $\mathfrak{p}$ is even and then it is well-known that L is the direct sum of two subspaces L^+ and L^- each of which is of dimension 2^{m-1} ($2m = \dim \mathfrak{p}$) and is stable under σ. We denote by σ^+ and σ^- the restrictions of σ to L^+ and L^- respectively. σ^+ and σ^- are called the half spin representations of $\mathrm{Spin}(\mathfrak{p})$. We now define

$$\chi^{\pm} = \sigma^{\pm} \circ \tilde{\alpha} \quad \text{and} \quad \chi = \sigma \circ \tilde{\alpha} \,. \tag{1.1}$$

Suppose now an irreducible unitary representation τ of K on a finite dimensional complex vector space V is given. Consider the representation $\chi \otimes \tau$ of K on $L \otimes V$. Because we have a principal K-bundle G on G/K, the representation $\chi \otimes \tau$ of K gives rise to a vector bundle $E_{L\otimes V}$ on G/K. (Given a representation τ: $K \to \mathrm{Aut}(W)$ where W is a vector space over $\mathbf{C}$ or $\mathbf{R}$, the induced bundle E_W on G/K, we recall, is defined as the quotient of $G \times W$ by the equivalence relation $(g, w) \sim (gk, k^{-1}w)$ for any $k \in K$, $g \in G$, and $w \in W$. The equivalence class of (g, w) will be denoted by $\{g, w\}$.) We now define the Dirac operator; it is a first order linear differential operator acting on the space $C(E_{L\otimes V})$ of C^{∞} sections of $E_{L\otimes V}$.

To define this we first recall certain basic facts concerning the spin group. Some of the material we have collected here can be found in [2] and [3]. Let $\mathfrak{m}$ be a real vector space with a positive definite inner product and $\mathrm{Cliff}(\mathfrak{m})$ the Clifford algebra of $\mathfrak{m}$. Thus $\mathrm{Cliff}(\mathfrak{m})$ is the quotient of the tensor algebra $T(\mathfrak{m})$ on $\mathfrak{m}$ by the ideal I generated by elements of the form $(x, x)\cdot 1 + x \otimes x$. The natural map $\mathfrak{m} \hookrightarrow T(\mathfrak{m}) \longrightarrow \mathrm{Cliff}(\mathfrak{m})$ is an inclusion by which we identify $\mathfrak{m}$ as a subspace of $\mathrm{Cliff}(\mathfrak{m})$. Then if $\{x_1, \cdots, x_n\}$ is any orthonormal base for $\mathfrak{m}$ one has the relations

$$x_i^2 = -1 \quad \text{and} \quad x_j x_k + x_k x_j = 0 \qquad (j \neq k) \,. \tag{1.2}$$

Also $\{x_{i_1} x_{i_2} \cdots x_{i_k} \,|\, 1 \leqq i_1 < i_2 < \cdots < i_k \leqq n\}$ is a base for $\mathrm{Cliff}(\mathfrak{m})$. $\mathrm{Cliff}(\mathfrak{m})$ has a $\mathbf{Z}_2$-graded structure $\mathrm{Cliff}(\mathfrak{m}) = C_+\mathfrak{m} + C_-\mathfrak{m}$ where $C_+\mathfrak{m}$ and $C_-\mathfrak{m}$ are respectively spanned by the even and odd products of elements in the above basis. One has $\mathfrak{m} \subseteqq C_-\mathfrak{m}$. The group $\mathrm{Spin}(n)$ (or $\mathrm{Spin}(\mathfrak{m})$) exists naturally as a subgroup of the group $C^*\mathfrak{m}$ of invertible elements of $\mathrm{Cliff}(\mathfrak{m})$. Indeed let $x \mapsto \bar{x}$ be the antiautomorphism of $\mathrm{Cliff}(\mathfrak{m})$ which is defined by

$$e_1 e_2 \cdots e_k \longmapsto (-1)^k e_k \cdots e_2 e_1$$

where $e_1, \cdots, e_k$ are arbitrary elements of $\mathfrak{m}$. (It can be checked easily that the map is well defined.) Then $\mathrm{Spin}(\mathfrak{m})$ is the subgroup of $C^*\mathfrak{m}$ consisting of all elements x such that

$$\begin{aligned} &x \in C_+\mathfrak{m} \\ &x e x^{-1} \in \mathfrak{m} \quad \text{for all } e \in \mathfrak{m} \\ &\bar{x}\cdot x = 1 \,. \end{aligned} \tag{1.3}$$

Thus for all $x \in \mathrm{Spin}(\mathfrak{m})$ the transformation $\psi(x)$: $\mathfrak{m} \to \mathfrak{m}$ defined by $e \mapsto x e x^{-1}$ is an orientation preserving isometry of $\mathfrak{m}$. Hence ψ maps $\mathrm{Spin}(\mathfrak{m})$ into $SO(\mathfrak{m})$. One knows that ψ is a twofold covering map onto $SO(\mathfrak{m})$. Moreover, $\mathrm{Spin}(\mathfrak{m})$ is simply connected if $n \geqq 3$.

When $\mathfrak{m}$ is even dimensional, the algebra $\mathrm{Cliff}(\mathfrak{m})$ as well as the algebra $\mathrm{Cliff}(\mathfrak{m}) \otimes \mathbf{C}$ is a simple algebra and hence $\mathrm{Cliff}(\mathfrak{m}) \otimes \mathbf{C}$ is isomorphic to the endomorphism algebra of a finite dimensional complex vector space L, through an isomorphism

$$\varepsilon\colon \mathrm{Cliff}(\mathfrak{m}) \otimes \mathbf{C} \longrightarrow \mathrm{End}(L)\ . \tag{1.4}$$

This isomorphism ε shall be fixed once for all. Since $\mathrm{Spin}(\mathfrak{m}) \subseteqq \mathrm{Cliff}(\mathfrak{m}) \otimes \mathbf{C}$, we have a restriction of ε to $\mathrm{Spin}(\mathfrak{m})$, denoted by σ. One knows that σ is the direct sum of two inequivalent sub-representations σ^+ and σ^- on subspaces L^+ and L^- of L each of which is of dimension 2^{m-1}, where $2m$ is the dimension of $\mathfrak{m}$. Let $\mathfrak{m}_2$ be the subspace of $\mathrm{Cliff}(\mathfrak{m})$ spanned by

$$\{x_{i_1}x_{i_2} \mid 1 \leqq i_1,\, i_2 \leqq 2m\ ,\ \ i_1 \neq i_2\}\ .$$

Note that

$$\{x_{i_1}x_{i_2} \mid 1 \leqq i_1 < i_2 \leqq 2m\} \tag{1.5}$$

is a base for $\mathfrak{m}_2$. For any element $z \in \mathrm{Cliff}(\mathfrak{m})$ let $X(z)\colon \mathrm{Cliff}(\mathfrak{m}) \to \mathrm{Cliff}(\mathfrak{m})$ be the linear map given by

$$x \longmapsto zx - xz \qquad\qquad (x \in \mathrm{Cliff}(\mathfrak{m}))\ .$$

An easy computation shows that for $1 \leqq i, j, k \leqq 2m$, $i \neq j$ one has

$$X(x_i x_j)(x_k) = \begin{cases} 0 & \text{if } k \neq i, j \\ 2x_j & \text{if } k = i \\ -2x_i & \text{if } k = j\ . \end{cases} \tag{1.6}$$

It follows that for $z \in \mathfrak{m}_2$, $X(z)(\mathfrak{m}) \subseteqq \mathfrak{m}$. Thus $X(z)$ restricts to an endomorphism, also denoted $X(z)$, of $\mathfrak{m}$. The following simple lemma is quite useful for us.

LEMMA 1.1. *Let* $\mathrm{Cliff}(\mathfrak{m})$ *be identified with a subalgebra of* $\mathrm{End}(\mathrm{Cliff}(\mathfrak{m}))$ *by the left regular representation*

$$l\colon \mathrm{Cliff}(\mathfrak{m}) \longrightarrow \mathrm{End}(\mathrm{Cliff}(m))\ .$$

When $\mathrm{End}(\mathrm{Cliff}(\mathfrak{m}))$ *is considered as the Lie algebra of* $\mathrm{Aut}(\mathrm{Cliff}(\mathfrak{m}))$ *in the usual way the subspace* $\mathfrak{m}_2$ *is the Lie subalgebra corresponding to the Lie subgroup* $\mathrm{Spin}(\mathfrak{m})$ *(i.e.* $l(\mathfrak{m}_2)$ *is the Lie subalgebra corresponding to the Lie subgroup* $l(\mathrm{Spin}(\mathfrak{m}))$ *of* $\mathrm{Aut}(\mathrm{Cliff}(\mathfrak{m}))$*). Now, given any algebra representation* $\sigma\colon \mathrm{Cliff}(\mathfrak{m}) \to \mathrm{End}(W)$, *(where* $\mathrm{End}(W)$ *is the endomorphism algebra of a finite dimensional real vector space* W *or the endomorphism algebra of a complex vector space* W*, considered as a real algebra), the differential of the group representation* $(\sigma \mid \mathrm{Spin}(\mathfrak{m}))\colon \mathrm{Spin}(\mathfrak{m}) \to \mathrm{Aut}(W)$ *is the Lie algebra representation* $(\sigma \mid \mathfrak{m}_2)\colon \mathfrak{m}_2 \to \mathrm{End}(W)$. *Moreover, the differential of the rep-*

resentation $\psi\colon \mathrm{Spin}(\mathfrak{m}) \to \mathrm{Aut}(\mathfrak{m})$ *is the representation* $X\colon \mathfrak{m}_2 \to \mathrm{End}(\mathfrak{m})$ (*namely* $z \mapsto X(z)$).

Proof. Note that for any $z \in \mathrm{Cliff}(\mathfrak{m})$, there exists a unique element of $\mathrm{Cliff}(\mathfrak{m})$ denoted by $\exp z$ such that $l(\exp z) = \exp l(z)$. The series $1 + z + z^2/2! + \cdots + z^k/k! + \cdots$ converges to $\exp z$. Now, let $z \in \mathfrak{m}_2$; we will show that $\exp z \in \mathrm{Spin}(\mathfrak{m})$. For this, in view of the defining relations (1.3), we have to show that $\exp z \in C_+\mathfrak{m}$, $\exp z \cdot \overline{(\exp z)} = 1$ and $\exp z \cdot e \cdot (\exp z)^{-1} \in \mathfrak{m}$ for $e \in \mathfrak{m}$. We have $\mathfrak{m}_2 \subseteqq C_+\mathfrak{m}$ and hence $\exp z = 1 + z + z^2/2! + \cdots \in C_+\mathfrak{m}$. Also, one sees that $\overline{(\exp z)} = 1 + \bar{z} + \bar{z}^2/2! + \cdots = \exp(\bar{z})$. But if $z = \sum a_{ij}x_i x_j$ is the expression for z in terms of the base (1.5) of $\mathfrak{m}_2$ then $\bar{z} = \sum a_{ij}x_j x_i = -\sum a_{ij}x_i x_j = -z$. Thus $\exp z \cdot \overline{(\exp z)} = \exp z \cdot \exp(-z) = 1$. Finally one sees easily that

$$\exp z \cdot e \cdot (\exp z)^{-1} = \exp\big(X(z)\big)(e) \in \mathfrak{m}$$

for $e \in \mathfrak{m}$, since $X(z)$ leaves $\mathfrak{m}$ stable. Thus $\exp z \in \mathrm{Spin}(\mathfrak{m})$ and we have

$$\psi(\exp z) = \exp X(z)\ . \tag{1.7}$$

It now follows that $l(\mathfrak{m}_2)$ is contained in the Lie subalgebra of $\mathrm{End}(\mathrm{Cliff}(\mathfrak{m}))$ corresponding to the Lie subgroup $l(\mathrm{Spin}(\mathfrak{m}))$ of $\mathrm{Aut}(\mathrm{Cliff}(\mathfrak{m}))$ (see [5], Prop. 2.7, p. 108). But both $\mathrm{Spin}(\mathfrak{m})$ and $\mathfrak{m}_2$ have dimension $2m(2m-1)/2$. Thus $l(\mathfrak{m}_2)$ is the Lie subalgebra of $l(\mathrm{Spin}(\mathfrak{m}))$, or identifying by l, $\mathfrak{m}_2$ is the Lie algebra of $\mathrm{Spin}(\mathfrak{m})$. Note that the Lie bracket $[z, z']$ of two elements $z, z' \in \mathfrak{m}_2$ is given by $[z, z'] = zz' - z'z$ and that the map $z \mapsto \exp z$ is the 'exponential' map of $\mathfrak{m}_2$ into $\mathrm{Spin}(\mathfrak{m})$. The other assertions of the lemma are elementary consequence of this fact. The last assertion, for example, follows using (1.7). (q.e.d.)

COROLLARY 1.1. *The differential of the spin representation*

$$\sigma\big(=\varepsilon\,|\,\mathrm{Spin}(\mathfrak{m})\big) : \mathrm{Spin}(\mathfrak{m}) \longrightarrow \mathrm{Aut}(L)$$

is the representation (*also denoted by* σ)

$$\sigma(=\varepsilon\,|\,\mathfrak{m}_2) : \mathfrak{m}_2 \longrightarrow \mathrm{End}(L)\ ,$$

where ε *is the representation of* $\mathrm{Cliff}(\mathfrak{m})$ *defined by* (1.4).

We now come to the definition of the Dirac operator D. We denote by $C^+(E_V)$ and $C^-(E_V)$ the spaces $C(E_{L^+\otimes V})$ and $C(E_{L^-\otimes V})$ of C^∞ sections of the bundles $E_{L^+\otimes V}$ and $E_{L^-\otimes V}$ which are induced by the representations $\chi^+ \otimes \tau$ and $\chi^- \otimes \tau$ respectively. Also, we denote by $C(E_{L\otimes V})$ the space of C^∞ sections of the bundle $E_{L\otimes V}$ on G/K induced by the representation $\chi \otimes \tau$ of K. We have natural inclusions

$$C^+(E_V) \longrightarrow C(E_{L\otimes V})$$

and

$$C^-(E_V) \longrightarrow C(E_{L\otimes V})$$

and we have $C(E_{L\otimes V}) = C^+(E_V) \oplus C^-(E_V)$. Observe that since L is a module for Cliff $\mathfrak{p} \otimes \mathbf{C}$ and since $\mathfrak{p}^{\mathbf{C}} \subseteq$ Cliff $\mathfrak{p} \otimes \mathbf{C}$ one has a natural bilinear pairing, also denoted by ε

$$\varepsilon\colon \mathfrak{p}^{\mathbf{C}} \otimes L \longrightarrow L \tag{1.8}$$

given by

$$\varepsilon(X \otimes s) = \varepsilon(X)(s) \qquad (X \in \mathfrak{p}^{\mathbf{C}},\ s \in L)\,. \tag{1.9}$$

This is a Spin($\mathfrak{p}$) module homomorphism, when $\mathfrak{p}^{\mathbf{C}} \otimes L$ is considered as a Spin($\mathfrak{p}$) module by the representation $\psi \otimes \sigma$. One knows that under this pairing $\mathfrak{p}^{\mathbf{C}} \otimes L^+$ maps into L^- and $\mathfrak{p}^{\mathbf{C}} \otimes L^-$ maps into L^+, thus giving rise to maps

$$\varepsilon\colon \mathfrak{p}^{\mathbf{C}} \otimes L^+ \longrightarrow L^-$$

and

$$\varepsilon\colon \mathfrak{p}^{\mathbf{C}} \otimes L^- \longrightarrow L^+\,.$$

Thus, we have a pairing of K modules

$$\begin{aligned} \varepsilon \otimes 1\colon \mathfrak{p}^{\mathbf{C}} \otimes L \otimes V &\longrightarrow L \otimes V \quad \text{and} \\ \varepsilon \otimes 1\colon \mathfrak{p}^{\mathbf{C}} \otimes L^{\pm} \otimes V &\longrightarrow L^{\mp} \otimes V\,. \end{aligned} \tag{1.10}$$

These maps induce bundle maps

$$\begin{aligned} \tilde{\mu}\colon E_{\mathfrak{p}^{\mathbf{C}}\otimes L\otimes V} &\longrightarrow E_{L\otimes V} \quad \text{and} \\ \tilde{\mu}^{\pm}\colon E_{\mathfrak{p}^{\mathbf{C}}\otimes L^{\pm}\otimes V} &\longrightarrow E_{L^{\mp}\otimes V}\,, \end{aligned} \tag{1.11}$$

where $E_{\mathfrak{p}^{\mathbf{C}}\otimes L\otimes V}, \cdots$ etc., denote the vector bundles on G/K induced by the representations of K on $\mathfrak{p}^{\mathbf{C}} \otimes L \otimes V, \cdots$ etc. We also denote by

$$\tilde{\mu}\colon C(E_{\mathfrak{p}^{\mathbf{C}}\otimes L\otimes V}) \longrightarrow C(E_{L\otimes V})$$

and

$$\tilde{\mu}^{\pm}\colon C(E_{\mathfrak{p}^{\mathbf{C}}\otimes L^{\pm}\otimes V}) \longrightarrow C(E_{L^{\mp}\otimes V})$$

the induced maps on C^∞ sections. On the other hand, the canonical connection (see [7(b), §2]) on the principal K-bundle G on G/K gives rise to connections on the induced bundles $E_{L\otimes V}$ and $E_{L^{\pm}\otimes V}$.

Remark: The representation α of K on $\mathfrak{p}^{\mathbf{C}}$ induces the complexified tangent bundle of G/K and the above connection on G gives rise to the Riemannian connection on $E_{\mathfrak{p}^{\mathbf{C}}}$. This is clear since the connection on G is torsionless and its holonomy group is contained in K. (See [7(b) Theorem 2.6 and Corollary 4.3].)

Identifying the (complexified) cotangent bundle of G/K with the tangent bundle $E_{\mathfrak{p}^C}$ of G/K by means of the complexification of the Riemannian metric, let

$$\nabla\colon C(E_{L\otimes V}) \longrightarrow C(E_{\mathfrak{p}^C\otimes L\otimes V})$$

and

$$\nabla^{\pm}\colon C(E_{L^{\pm}\otimes V}) \longrightarrow C(E_{\mathfrak{p}^C\otimes L^{\pm}\otimes V})$$

be the covariant differentiation associated to the above connections. Recalling that $C^{\pm}(E_V)$ is our earlier notation for $C(E_{L^{\pm}\otimes V})$, we now have the Dirac operators

$$D\colon C(E_{L\otimes V}) \longrightarrow C(E_{L\otimes V})$$

and

$$D^{\pm}\colon C^{\pm}(E_V) \longrightarrow C^{\mp}(E_V)$$

defined respectively by

$$D = \tilde{\mu}\circ\nabla \quad\text{and}\qquad D^{\pm} = \tilde{\mu}^{\pm}\circ\nabla^{\pm}\ . \tag{1.12}$$

We remark that these operators are elliptic; this is because for any $X \neq 0$ in $\mathfrak{p}$, the maps $\varepsilon(X)\colon L\otimes V \to L\otimes V$ and $\varepsilon(X)\colon L^{\pm}\otimes V\to L^{\mp}\otimes V$ are isomorphisms and D, $D^{\pm}$ are homogeneous.

For defining the Dirac operator we have followed [2].

We now want to describe an explicit formula for the Dirac operator. For this, we define a representation ν of $\mathfrak{g}^C$ on $C^{\infty}(G)$, the space of complex valued infinitely differentiable functions on G, by

$$\nu(X)f = Xf \qquad \left(X\in\mathfrak{g}^C\ ,\ \ f\in C^{\infty}(G)\right)\ .$$

Let $U(\mathfrak{g}^C)$ be the universal enveloping algebra of $\mathfrak{g}^C$. Then ν defines a representation, also denoted by ν, of $U(\mathfrak{g}^C)$ on $C^{\infty}(G)$.

We now define

$$C^{+}(G,\ V) = L^{+}\otimes C^{\infty}(G)\otimes V \quad\text{and}\qquad C^{-}(G,\ V) = L^{-}\otimes C^{\infty}(G)\otimes V\ . \tag{1.13}$$

We define an injective mapping

$$\eta\colon C(E_{L\otimes V}) \longrightarrow L\otimes C^{\infty}(G)\otimes V \tag{1.14}$$

as follows: Let φ be a section of $E_{L\otimes V}$. Since $E_{L\otimes V}$ is induced from the principal K-bundle G, we have a canonical map

$$G\times(L\otimes V) \longrightarrow E_{L\otimes V}$$

denoted

$$(g, z) \longmapsto \{g, z\}\ , \qquad (g \in G\ ,\ \ z \in L \otimes V)\ .$$

This gives rise to an $L \otimes V$ valued C^∞ function $\eta(\varphi)$ on G defined by

$$\{g, \eta(\varphi)(g)\} = \varphi(gK)\ .$$

Identifying the space of $L \otimes V$ valued C^∞ functions on G with $L \otimes C^\infty(G) \otimes V$ in the usual way we get the desired element $\eta(\varphi) \in L \otimes C^\infty(G) \otimes V$. It is clear that η maps $C^+(E_V)$ into $C^+(G, V)$ and $C^-(E_V)$ into $C^-(G, V)$. Put

$$(1.15)\qquad \begin{aligned} &\big(L \otimes C^\infty(G) \otimes V\big)^0 \\ &= \{u \in L \otimes C^\infty(G) \otimes V \mid (\chi \otimes r \otimes \tau)(k)(u) = u,\ (k \in K)\} \end{aligned}$$

where r denotes the right representation of G on $C^\infty(G)$. Put also

$$(1.16)\qquad \begin{aligned} C^+(G, V)^0 &= C^+(G, V) \cap \big(L \otimes C^\infty(G) \otimes V\big)^0 \\ C^-(G, V)^0 &= C^-(G, V) \cap \big(L \otimes C^\infty(G) \otimes V\big)^0\ . \end{aligned}$$

Then η maps $C(E_{L\otimes V})$ and $C^\pm(E_V)$ isomorphically onto $\big(L \otimes C^\infty(G) \otimes V\big)^0$ and $C^\pm(G, V)^0$ respectively. We now have the following

PROPOSITION 1.1. *For* $u \in C(E_{L\otimes V})$

$$\eta(Du) = \big(\textstyle\sum_{i=1}^{2m} \varepsilon(X_i) \otimes \nu(X_i) \otimes 1\big)(\eta u)\ ,$$

where $X_1, \cdots, X_{2m}$ *is any orthonormal base of* $\mathfrak{p}$ *and where* $\varepsilon(X_i)$ *denotes the action of* $X_i \in \mathfrak{p} \subseteqq \mathrm{Cliff}(\mathfrak{p})$ *on the* $\mathrm{Cliff}(\mathfrak{p})$ *module* L.

Proof. The proof of this proposition is straightforward and is left to the reader.

2. Some notation and two lemmas

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$. Let Σ be the set of all nonzero roots of $\mathfrak{g}^C$ with respect to $\mathfrak{h}^C$. A root $\alpha \in \Sigma$ is called compact if the corresponding root space is contained in $\mathfrak{k}^C$ and noncompact if the root space is contained in $\mathfrak{p}^C$. Let $W = W(\mathfrak{h}^C, \mathfrak{g}^C)$ be the Weyl group of $\mathfrak{g}^C$ and let W_G be the subgroup of W generated by reflections with respect to compact roots. We denote by P the set of all positive roots, by P_k the set of all compact positive roots and by P_n the set of all noncompact positive roots with respect to a lexicographic ordering in Σ, which we fix once for all. Then we have $P = P_k \cup P_n$. We define

$$\rho = 1/2 \textstyle\sum_{\alpha \in P} \alpha\ , \quad \rho_k = 1/2 \sum_{\alpha \in P_k} \alpha, \quad \text{and} \quad \rho_n = 1/2 \sum_{\alpha \in P_n} \alpha\ .$$

In the dual $(\mathfrak{h}^C)'$ of $\mathfrak{h}^C$ we introduce the bilinear form $\langle\ ,\ \rangle$ induced by the Killing form B of $\mathfrak{g}^C$, in the usual way.

We now study the representation χ of §1 in more detail. As remarked in §1, the homomorphism $\alpha: K \to SO(\mathfrak{p})$ may not be liftable to a homomorphism $\tilde{\alpha}: K \to \mathrm{Spin}(\mathfrak{p})$. However, since $\mathrm{Spin}(\mathfrak{p}) \to SO(\mathfrak{p})$ is a two-fold covering, there exists a covering $K_1 \to K$, which is either trivial or two-fold, such that $\alpha: K \to SO(\mathfrak{p})$ lifts to a homomorphism $\tilde{\alpha}: K_1 \to \mathrm{Spin}(\mathfrak{p})$. The inclusion $K \subset G$ induces an isomorphism of fundamental groups. Hence, $K_1 \to K$ extends to a covering $G_1 \to G$ which is again either trivial or two-fold, such that $K_1 \subset G_1$ corresponds to the subalgebra $\mathfrak{k}$.

Remark 2.1. Let $\mathfrak{t}$ be a Cartan subalgebra of the Lie algebra of $SO(\mathfrak{p})$, chosen so that $\alpha(\mathfrak{h}) \subseteq \mathfrak{t}$. Since $\mathfrak{p}$ is even dimensional, the weights of the standard representation of $SO(\mathfrak{p})$ are of the form $\pm\lambda_1, \pm\lambda_2, \cdots, \pm\lambda_m$. It is known that in these terms, the sets $\Delta^\pm$ of weights of $\sigma^\pm$ are given by

$$\Delta^+ = \{1/2(\pm\lambda_1 \pm\lambda_2 \pm \cdots \pm\lambda_m) \mid \text{with an even number of negative signs occurring}\}$$

and

$$\Delta^- = \{1/2(\pm\lambda_1 \pm\lambda_2 \pm \cdots \pm\lambda_m) \mid \text{with an odd number of negative signs occurring}\}.$$

By the definition of α, α composed with the standard representation of $SO(\mathfrak{p})$ is precisely the adjoint representation of K on $\mathfrak{p}$. Hence, α^* is a bijection between $\pm\lambda_1, \pm\lambda_2, \cdots, \pm\lambda_m$ and the set of noncompact roots. It may be assumed that $\alpha^*\lambda_i = \alpha_i$, where $\alpha_1, \cdots, \alpha_m$ is an enumeration of the positive noncompact roots. Since $\chi^\pm = \sigma^\pm \circ \alpha$, it follows that the sets of weights of χ^+ and χ^- are respectively given by

$$\{1/2(\pm\alpha_1 \pm\alpha_2 \pm \cdots \pm\alpha_m) \mid \text{with an even number of negative signs occurring}\} \tag{2.1}$$

and

$$\{1/2(\pm\alpha_1 \pm\alpha_2 \pm \cdots \pm\alpha_m) \mid \text{with an odd number of negative signs occurring}\}. \tag{2.2}$$

The multiplicity of each weight is the number of ways in which it can be expressed in the above form. In particular, we note that $\rho_n = 1/2(\alpha_1 + \cdots + \alpha_m)$ is a weight of the representation χ. In fact ρ_n is the highest weight of an irreducible component of χ. For this, first observe that the weights $1/2(\pm\alpha_1 \pm \cdots \pm \alpha_m)$ are just the elements $\rho_n - \langle\Phi\rangle$ where Φ runs through subsets of P_n and $\langle\Phi\rangle =$ the sum of elements in Φ. Now, if ρ_n is not a highest weight we would have $\rho_n + \beta = \rho_n - \langle\Phi\rangle$ where β is a positive compact root and Φ is a subset of P_n. But this is a contradiction, for this implies $\beta = -\langle\Phi\rangle =$ a nonpositive integral linear combination of simple roots in P.

Remark 2.2. From the above explicit description of weights of χ one can easily see that if $X \in \mathfrak{h}$, then

$$\operatorname{Trace}\left(\chi^{+}(\exp X)\right) - \operatorname{Trace}\left(\chi^{-}(\exp X)\right) = \prod_{\alpha \in P_n} \left(e^{\alpha(X)/2} - e^{-\alpha(X)/2}\right).$$

Now, we give an explicit formula for the differential $\tilde{\alpha}: \mathfrak{k} \to \mathfrak{p}_2$ of the homomorphism $\tilde{\alpha}: K_1 \to \operatorname{Spin}(\mathfrak{p})$, where $\mathfrak{p}_2 \subseteqq \operatorname{Cliff}(\mathfrak{p})$ is considered as the Lie algebra of $\operatorname{Spin}(\mathfrak{p})$ by Lemma 1.1.

LEMMA 2.1. *Let* $Y \in \mathfrak{k}$. *Then,* $\tilde{\alpha}(Y) \in \mathfrak{p}_2$ *is given by*

$$\tilde{\alpha}(Y) = \sum_{k,l=1}^{2m} \frac{B([Y, X_k], X_l)}{4} X_k X_l,$$

where $X_1, \cdots, X_{2m}$ *is any base of* $\mathfrak{p}$, *chosen so that* $B(X_i, X_j) = \delta_{ij}$.

Proof. For any $Y \in \mathfrak{k}$, we have

$$X(\tilde{\alpha}(Y)) = \operatorname{ad} Y. \tag{2.3}$$

Recall we have the relations (see 1.6)

$$X(X_i X_j)(X_k) = \begin{cases} 0 & \text{if } k \neq i, j \\ 2X_j & \text{if } k = i \\ -2X_i & \text{if } k = j \end{cases} \qquad (1 \leqq i, j, k \leqq 2m, \quad i \neq j). \tag{2.4}$$

Now, writing

$$\tilde{\alpha}(Y) = \sum_{\substack{k,l=1 \\ k<l}}^{2m} C_{kl} X_k X_l,$$

we have, using (2.3)

$$[Y, X_i] = \sum_{k<l} C_{kl} X(X_k X_l)(X_i) \qquad (i = 1, \cdots, 2m).$$

Hence, it is clear using (2.4) that

$$B([Y, X_i], X_j) = 2C_{ij} \qquad (i < j).$$

Thus, we have

$$\tilde{\alpha}(Y) = \sum_{k<l} \frac{B([Y, X_k], X_l)}{2} X_k X_l.$$

Now, the lemma follows on observing that

$$B([Y, X_k], X_k) = 0 \qquad (k = 1, \cdots, 2m)$$

and

$$B([Y, X_k], X_l) X_k X_l = B([Y, X_l], X_k) X_l X_k. \qquad \text{(q.e.d.)}$$

We define a subset W^1 of W by

$$W^1 = \{\sigma \in W \mid \sigma P \supset P_k\}. \tag{2.5}$$

Remark 2.3. One can easily prove that the map $W_G \times W^1 \to W$ given by $(s, \sigma) \mapsto s\sigma$ is a bijection. Also, for any $\sigma \in W^1$ if $P_n^{(\sigma)}$ is the set of noncompact roots in σP and if

$$\rho_n^{(\sigma)} = 1/2 \sum_{\alpha \in P_n^{(\sigma)}} \alpha$$

then as in Remark 2.1 one can show that $\rho_n^{(\sigma)}$ is the highest weight of an irreducible component of χ. Note that $\rho_n^{(\sigma)} = \sigma\rho - \rho_k$.

Now, let $Y_1, Y_2, \cdots, Y_t$ be any base of $\mathfrak{k}$ chosen so that $B(Y_i, Y_j) = -\delta_{ij}$. We then have the following

LEMMA 2.2. *For the representation χ of $\mathfrak{k}$, the Casimir $-Y_1^2 - Y_2^2 - \cdots - Y_t^2$ of $\mathfrak{k}$ acts as scalar multiplication by $\langle\rho, \rho\rangle - \langle\rho_k, \rho_k\rangle$. For any $\sigma \in W^1$, if $\tau_{\sigma\rho-\rho_k}$ denotes the irreducible representation of $\mathfrak{k}$ with highest weight $\sigma\rho - \rho_k$, then $\tau_{\sigma\rho-\rho_k}$ occurs with multiplicity one in χ and we have*

$$\chi^+ = \bigoplus_{\sigma \in W^1, \varepsilon(\sigma)=+1} \tau_{\sigma\rho-\rho_k}$$

and

$$\chi^- = \bigoplus_{\sigma \in W^1, \varepsilon(\sigma)=-1} \tau_{\sigma\rho-\rho_k} .$$

Proof. First observe that the sets of weights of χ^+ and χ^- do not intersect. Indeed, if this were not true, there would have to be an identity

$$\rho_n - \alpha_1 - \alpha_2 - \cdots - \alpha_k = \rho_n - \alpha_1' - \alpha_2' - \cdots - \alpha_m'$$

where the α's are positive noncompact roots, with k even and m odd. When a noncompact positive root is expressed as an integral linear combination of simple roots, the sum of the coefficients of all the noncompact ones among the simple roots are odd. This contradicts the identity above. It follows that χ^+ and χ^- cannot have an irreducible component in common. By Remark 2.2, we have on H_1, the subgroup of G_1 corresponding to $\mathfrak{h} \subset \mathfrak{g}$,

$$\begin{aligned}\text{Trace}\,\chi^+ - \text{Trace}\,\chi^- &= \prod_{\alpha \in P_n} (e^{\alpha/2} - e^{-\alpha/2}) \\ &= \prod_{\alpha \in P} (e^{\alpha/2} - e^{-\alpha/2}) \prod_{\alpha \in P_k} (e^{\alpha/2} - e^{-\alpha/2})^{-1} \\ &= \sum_{s \in W} \varepsilon(s) e^{s\rho} \Big(\sum_{s \in W_G} \varepsilon(s) e^{s\rho_k}\Big)^{-1} \\ &= \sum_{\sigma \in W^1} \varepsilon(\sigma) \Big(\sum_{s \in W_G} \varepsilon(s) e^{s\sigma\rho}\Big) \Big(\sum_{s \in W_G} \varepsilon(s) e_s{}^{\rho_k}\Big)^{-1} .\end{aligned}$$

Note that for $\sigma \in W^1$, $\sigma\rho$ is dominant with respect to P_k. Since χ^+ and χ^- have no irreducible components in common, the lemma now follows using Weyl's character formula and observing that on the K-module $\tau_{\sigma\rho-\rho_k}$ the Casimir of K acts as scalar multiplication by $\langle\sigma\rho, \sigma\rho\rangle - \langle\rho_k, \rho_k\rangle = \langle\rho, \rho\rangle - \langle\rho_k, \rho_k\rangle$. (q.e.d.)

Remark. The fact that for the representation χ *of* $\mathfrak{k}$ the Casimir $\Omega_k (= -Y_1^2 - Y_2^2 - \cdots - Y_t^2)$ acts as scalar multiplication by a constant can be proved even without the assumption that rank of $\mathfrak{k}$ = rank of $\mathfrak{g}$.

3. A formula for the operator $\square$

Let τ be an irreducible representation of K_1 on V. Consider the Dirac operator

$$D\colon C(E_{L\otimes V}) \longrightarrow C(E_{L\otimes V})$$

and

$$D^{\pm}\colon C^{\pm}(E_V) \longrightarrow C^{\mp}(E_V) .$$

We now define an operator $\square\colon C(E_{L\otimes V}) \to C(E_{L\otimes V})$ by

$$\square = D^2 . \tag{3.1}$$

Note that one has $\square\big(C^{\pm}(E_V)\big) \subseteqq C^{\pm}(E_V)$. Also, note that $\square\,|\,C^{+}(E_V) = D^{-}\circ D^{+}$ and $\square\,|\,C^{-}(E_V) = D^{+}\circ D^{-}$.

We now fix a base $\{Y_1, \cdots, Y_t\}$ of $\mathfrak{k}$ and a base $\{X_1, \cdots, X_{2m}\}$ of $\mathfrak{p}$, such that

$$B(Y_i,\ Y_j) = -\,\delta_{ij} \quad \text{and} \qquad B(X_i,\ X_j) = \delta_{ij} . \tag{3.2}$$

Note that $B(Y_i, X_j) = 0$. We then have the Casimir operator $\Omega = -Y_1^2 - \cdots - Y_t^2 + X_1^2 + \cdots + X_{2m}^2 \in U(\mathfrak{g}^{C})$.

PROPOSITION 3.1. *If λ is the highest weight of τ, then the operator $\square$ is given by*

$$\eta(\square u) = \{-1\otimes \nu(\Omega)\otimes 1 + \langle \lambda + \rho_n + 2\rho_k,\ \lambda - \rho_n\rangle\}\eta(u) \qquad \big(u\in C(E_{L\otimes V})\big) .$$

Here η is the map $C(E_{L\otimes V}) \to L\otimes C^{\infty}(G)\otimes V$ defined in (1.14) *and ν is the action of $U(\mathfrak{g}^{C})$ on $C^{\infty}(G_1)$ defined in* §1.

Proof. We recall that η is an injection of $C(E_{L\otimes V})$ onto $\big(L\otimes C^{\infty}(G_1)\otimes V\big)^0$ where the equivariant part $\big(L\otimes C^{\infty}(G_1)\otimes V\big)^0$ is defined by

$$\big(L\otimes C^{\infty}(G_1)\otimes V\big)^0 = \{\varphi\in L\otimes C^{\infty}(G_1)\otimes V \mid \big(\chi(k)\otimes r(k)\otimes \tau(k)\big)(\varphi) = \varphi \ \text{ for all } k\in K_1\}$$

where χ is the composite of the map $\tilde{\alpha}\colon K_1 \to \mathrm{Spin}(\mathfrak{p})$ with the spin representation $\sigma\colon \mathrm{Spin}(\mathfrak{p}) \to \mathrm{Aut}(L)$ and r the right regular representation of G_1 on $C^{\infty}(G_1)$.

Equivalently, we have,

$$\big(L\otimes C^{\infty}(G_1)\otimes V\big)^0 = \{\varphi\in L\otimes C^{\infty}(G_1)\otimes V \mid \big(\chi(Y)\otimes 1\otimes 1 + 1\otimes \nu(Y)\otimes 1 + 1\otimes 1\otimes \tau(Y)\big)\varphi = 0 ,\ \text{for all } Y\in \mathfrak{k}\} , \tag{3.3}$$

where, for convenience, we have denoted the differentials of χ and τ by the same letters.

By Proposition 1.1, to compute $\square$ on $C(E_{L\otimes V})$, we have only to compute

$$\left(\sum_{i=1}^{2m} \varepsilon(X_i) \otimes \nu(X_i) \otimes 1\right)^2 \big| (L \otimes C^\infty(C_1) \otimes V)^0 .$$

We have

$$\begin{aligned}
&\left(\sum_{i=1}^{2m} \varepsilon(X_i) \otimes \nu(X_i) \otimes 1\right)^2 \\
(3.4) \qquad &= \sum_{i=1}^{2m} \varepsilon(X_i)^2 \otimes \nu(X_i)^2 \otimes 1 + \sum_{i\neq j} \varepsilon(X_i)\varepsilon(X_j) \otimes \nu(X_i)\nu(X_j) \otimes 1 \\
&= \sum_{i=1}^{2m} - 1 \otimes \nu(X_i)^2 \otimes 1 + \frac{1}{2}\sum_{i,j} \varepsilon(X_i)\varepsilon(X_j) \otimes \nu[X_i, X_j] \otimes 1 ,
\end{aligned}$$

since, by the relations (1.2) we have

$$\varepsilon(X_1)^2 = - \mathrm{Id}, \quad \varepsilon(X_i)\varepsilon(X_j) = - \varepsilon(X_j)\varepsilon(X_i) \qquad (i \neq j) .$$

Consider the second term on the right hand side of (3.4). Because of (3.2), we have

$$[X_i, X_j] = - \sum_{q=1}^{t} B([X_i, X_j], Y_q) Y_q .$$

Thus,

$$\begin{aligned}
&\frac{1}{2} \sum_{i,j} \varepsilon(X_i)\varepsilon(X_j) \otimes \nu([X_i, X_j]) \otimes 1 \\
&= - \frac{1}{2} \sum_{q=1}^{t} \sum_{i,j} B([X_i, X_j], Y_q)\varepsilon(X_i)\varepsilon(X_j) \otimes \nu(Y_q) \otimes 1 \\
&= - \frac{1}{2} \sum_{q=1}^{t} \sum_{i,j} B([Y_q, X_i], X_j)\varepsilon(X_i)\varepsilon(X_j) \otimes \nu(Y_q) \otimes 1 \\
&= - 2 \sum_{q=1}^{t} \chi(Y_q) \otimes \nu(Y_q) \otimes 1 \qquad \text{(by Lemma 2.1)} \\
&= - \sum_q (\chi \otimes \nu)(Y_q)^2 \otimes 1 + \sum_q \chi(Y_q)^2 \otimes 1 \otimes 1 + \sum_q 1 \otimes \nu(Y_q)^2 \otimes 1 .
\end{aligned}$$

Thus (3.4) becomes

$$\begin{aligned}
&\left(\sum_i \varepsilon(X_i) \otimes \nu(X_i) \otimes 1\right)^2 \\
&= - 1 \otimes \nu(\Omega) \otimes 1 + (\chi \otimes \nu)(\Omega_K) \otimes 1 - \chi(\Omega_K) \otimes 1 \otimes 1 .
\end{aligned}$$

Via $\chi \otimes \nu \otimes \tau$, K operates trivially on $(L \otimes C^\infty(G) \otimes V)^0$. Thus on this space, $(\chi \otimes \nu)(\Omega_K) \otimes 1 = 1 \otimes 1 \otimes \tau(\Omega_K)$. According to Lemma 2.2, $\chi(\Omega_K) = \langle\rho, \rho\rangle 1 - \langle\rho_k, \rho_k\rangle 1$. Since λ is the highest weight of τ, $\tau(\Omega_K) = \langle\lambda + 2\rho_k, \lambda\rangle 1$. Together these facts yield

$$\begin{aligned}
&\left(\sum_i \varepsilon(X_i) \otimes \nu(X_i) \otimes 1\right)^2 \big| (L \otimes C^\infty(G) \otimes V)^0 \\
&= - 1 \otimes \nu(\Omega) \otimes 1 + \{\langle\lambda + 2\rho_k, \lambda\rangle - \langle\rho, \rho\rangle + \langle\rho_k, \rho_k\rangle\}\cdot 1 \\
&= - 1 \otimes \nu(\Omega) \otimes 1 + \langle\lambda + \rho_n + 2\rho_k, \lambda - \rho_n\rangle\cdot 1 .
\end{aligned}$$

Remark 3.1. Consider the representation T of G_1 on $L \otimes C^\infty(G_1) \otimes V$

given by $T_g = 1 \otimes l(g) \otimes 1$, where l denotes the left regular representation of G_1. Then it is easy to see that the canonical action $\hat{\pi}$ of G_1 on $C(E_{L\otimes V})$ given by

$$\big(\hat{\pi}(g)(s)\big)(g'K) = g \cdot s(g^{-1}g'K)$$

goes over by means of the identification map η into T_g on $\big(L \otimes C^\infty(G_1) \otimes V\big)^0$. Again under this identification it is easy to see that the action of an element $X \in \mathfrak{g}$ on $C(E_{L\otimes V})$ corresponds to the action on $\big(L \otimes C^\infty(G_1) \otimes V\big)^0$ which is the restriction of the action on $L \otimes C^\infty(G_1) \otimes V$ given by

$$x \otimes f \otimes v \longmapsto -x \otimes X^0 f \otimes v ,$$

where X^0 is the right invariant vector field on G_1 which has the same value at e as X. Thus the action $\hat{\pi}(\Omega)$ of the Casimir of $\mathfrak{g}$ corresponds to the restriction to $\big(L \otimes C^\infty(G_1) \otimes V\big)^0$ of the action

$$x \otimes f \otimes v \longmapsto x \otimes (-Y_1^{02} - Y_2^{02} - \cdots - Y_2^{02} + X_1^{02} + \cdots + X_{2m}^{02}) f \otimes v$$

which is the same as

$$x \otimes f \otimes v \longmapsto x \otimes (-Y_1^2 - Y_2^2 - \cdots - Y_t^2 + X_1^2 + \cdots + X_{2m}^2) f \otimes v$$

i.e.

$$x \otimes f \otimes v \longmapsto \big(1 \otimes \nu(\Omega) \otimes 1\big)(x \otimes f \otimes v) .$$

Thus the action $\hat{\pi}(\Omega)$ of the Casimir of $\mathfrak{g}$ on $C(E_{L\otimes V})$ corresponds to the action $1 \otimes \nu(\Omega) \otimes 1$ on $\big(L \otimes C^\infty(G_1) \otimes V\big)^0$.

We now define a subset $\mathcal{F}_0'$ of Hom $(\mathfrak{h}^C, C)$, the significance of which will become clear in §5. Let $\mathcal{F}$ be the set of all integral linear forms on $\mathfrak{h}^C$; i.e.

$$\mathcal{F} = \{\lambda \in \mathrm{Hom}\,(\mathfrak{h}^C, C) \; ; \; 2\langle\lambda, \alpha\rangle/\langle\alpha, \alpha\rangle \in \mathbf{Z} , \;\; \forall \alpha \in P\} .$$

We put

$$\mathcal{F}' = \{\lambda \in \mathcal{F} \; ; \; \langle\lambda + \rho, \alpha\rangle \neq 0 \;\; \forall \alpha \in P\}$$

and

$$\mathcal{F}_0' = \{\lambda \in \mathcal{F}' \; ; \; \langle\lambda + \rho, \alpha\rangle > 0 \;\; \forall \alpha \in P_k\} .$$

One sees that if $\lambda \in \mathcal{F}'$, then $\lambda \in \mathcal{F}_0'$ if and only if $\lambda + \rho_n$ is dominant with respect to P_k (i.e., $\langle\lambda + \rho_n, \alpha\rangle \geqq 0$ for all $\forall \alpha \in P_k$).

From now on we make the further assumption that the complexification G^C of G is simply connected. This assumption is made only to avoid the notational inconvenience that one runs into otherwise. In Remark 8.2, we explain how the constructions in the later sections go through without this assumption.

Now choose $\lambda \in \mathcal{F}_0'$. By our choice of G, λ gives rise to a character on the torus $H \subset G$ and hence also on $H_1 \subset G_1$, where H_1 corresponds to $\mathfrak{h} \subset \mathfrak{g}$. Also, as we saw in Remark 2.1, ρ_n gives rise to a character on H_1. Thus we have an irreducible representation $\lambda + \rho_n$ of K_1 on $V_{\lambda+\rho_n}$ with highest weight $\lambda + \rho_n$. Now consider the bundle $E_{L\otimes V_{\lambda+\rho_n}}$ on $G/K = G_1/K_1$ induced by the representation $\chi \otimes \tau_{\lambda+\rho_n}$ of K_1 on $L \otimes V_{\lambda+\rho_n}$.

Remark 3.2. We assert that the action $\tilde{\pi}$ of G_1 on $C(E_{L\otimes V_{\lambda+\rho_n}})$ goes down to an action also denoted $\tilde{\pi}$ of G on $C(E_{L\otimes V_{\lambda+\rho_n}})$. For this, it is enough to prove that the action of K_1 on $L \otimes V_{\lambda+\rho_n}$ goes down to an action of K on $L \otimes V_{\lambda+\rho_n}$. Now, with respect to $\mathfrak{h}$, any weight of the representation $L \otimes V_{\lambda+\rho_n}$ is of the form $\mu + \nu$, where μ is a weight of L and ν a weight of $V_{\lambda+\rho_n}$. But one knows that ν is of the form $\lambda + \rho_n - \sum m_i\alpha_i$ where $\sum m_i\alpha_i$ is a non-negative integral linear combination of positive compact roots. Also, by Remark 2.1, μ is of the form $\rho_n - \langle Q\rangle$, where $Q \subseteq P_n$ and $\langle Q\rangle = \sum_{\alpha\in Q}\alpha$. Thus, $\mu + \nu$ is of the form $\lambda + 2\rho_n - \sum n_i\alpha_i$, where $\sum n_i\alpha_i$ is a non-negative integral linear combination of positive roots. But, λ, $2\rho_n$, and $\sum n_i\alpha_i$ give rise to characters on H. Hence each weight of the representation $L \otimes V_{\lambda+\rho_n}$ gives rise to a character on the torus H. Hence the representation $L \otimes V_{\lambda+\rho_n}$ of K_1 is actually a representation of K.

Now consider the Dirac operator D and the operator $\square = D^2$ on $C(E_{L\otimes V_{\lambda+\rho_n}})$. Now, from Proposition 3.1 and Remark 3.2, we have the following

PROPOSITION 3.2. *Let $\lambda \in \mathcal{F}'_0$ and $V_{\lambda+\rho_n}$ the irreducible representation of K_1 with highest weight $\lambda + \rho_n$. Let $\tilde{\pi}$ denote the action G on $C(E_{L\otimes V_{\lambda+\rho_n}})$. For the derived action of the universal enveloping algebra of* $\mathfrak{g}$, *if $\tilde{\pi}(\Omega)$ denotes the action of the Casimir operator, then the operator $\square$ on $C(E_{L\otimes V_{\lambda+\rho_n}})$ is given by*

$$\square(u) = \{-\tilde{\pi}(\Omega) + \langle\lambda + 2\rho, \lambda\rangle\}(u) .$$

4. A consequence of the completeness of the metric on G/K

In this section we study the operator $\square$ in more detail. The groups G_1 and G will be as in §2. We fix a G_1 invariant measure in $G_1/K_1(=G/K)$ and a Haar measure in G_1 such that for $f \in C_c^\infty(G_1/K_1)$

$$\int_{G_1/K_1} f\,dx = \int_{G_1} \eta(f)dg \tag{4.1}$$

where $\eta(f)$ denotes the composite function $G_1 \to G_1/K_1 \xrightarrow{f} \mathbf{C}$. Let an irreducible representation τ of K_1 on V be given. We will define an inner product $(\ ,\ )$ in $C_c(E_{L\otimes V})$ where $C_c(E_{L\otimes V}) = \{u \in C(E_{L\otimes V}) \mid u$ has compact support$\}$. First, we prove the following

LEMMA 4.1. *There exists a positive definite hermitian metric on L such that for every $X \in \mathfrak{p}$ the symbol map $\varepsilon(X): L \to L$ leaves the hermitian metric infinitesimally invariant, i.e.,*

$$\big(\varepsilon(X)\, l,\, l'\big) + \big(l,\, \varepsilon(X)\, l'\big) = 0$$

for every $l,\ l' \in L$.

Proof. Let $X_1, \cdots, X_{2m}$ be an orthonormal base for $\mathfrak{p}$. Let $\mathfrak{q} = \mathfrak{p} \oplus \mathbf{R}$ and let a positive definite bilinear form on $\mathfrak{q}$ be chosen such that for some $X_{2m+1} \in \mathbf{R}$, $X_1, \cdots, X_{2m}, X_{2m+1}$ is an orthonormal base for $\mathfrak{q}$. Let Cliff($\mathfrak{q}$) be the (real) Clifford algebra on $\mathfrak{q}$ so that one has $X_i^2 = -1$ and $X_iX_j + X_jX_i = 0$ $(1 \leqq i, j \leqq 2m+1,\ i \neq j)$. Consider the map $\varphi: \mathfrak{p} \to \mathrm{Cliff}(\mathfrak{q})$ given by $X_i \mapsto X_iX_{2m+1}$ $(1 \leqq i \leqq 2m)$. One checks easily that $(X_iX_{2m+1})^2 = -1$ and $(X_iX_{2m+1})(X_jX_{2m+1}) + (X_jX_{2m+1})(X_iX_{2m+1}) = 0$ $(i \neq j)$. Thus φ extends to a homomorphism of the algebra Cliff($\mathfrak{p}$) into Cliff($\mathfrak{q}$). Since Cliff($\mathfrak{p}$) has no nontrivial two sided ideal it is easy to see that $\varphi: \mathrm{Cliff}(\mathfrak{p}) \to \mathrm{Cliff}(\mathfrak{q})$ is an injection. Clearly, $\varphi(\mathfrak{p} + \mathfrak{p}_2) \subseteqq \mathfrak{q}_2$ where $\mathfrak{q}_2$ is the subspace of Cliff($\mathfrak{q}$) generated by $\{X_iX_j \mid 1 \leqq i, j \leqq 2m+1,\ i \neq j\}$. But, by Lemma 1.1, $\mathfrak{q}_2$ is isomorphic to the Lie algebra of the rotation group $SO(\mathfrak{q})$. Note that the spaces $\mathfrak{p} + \mathfrak{p}_2$ and $\mathfrak{q}_2$ have the same dimension. Since $\varphi: \mathrm{Cliff}(\mathfrak{p}) \to \mathrm{Cliff}(\mathfrak{q})$ is an injection, it now follows that $\mathfrak{p} + \mathfrak{p}_2$ is a Lie subalgebra of Cliff($\mathfrak{p}$) (considered as a Lie algebra under commutation) isomorphic to the Lie algebra of $SO(\mathfrak{q})$ which is a compact Lie algebra. It now follows that there exists a positive definite hermitian metric on L which is infinitesimally invariant under $\varepsilon(X): L \to L$ for all $X \in \mathfrak{p} + \mathfrak{p}_2$.

(q.e.d.)

We now fix a hermitian metric (,) on L such that for every $X \in \mathfrak{p} + \mathfrak{p}_2$ and $u, v \in L$ one has

$$(4.2)\qquad \big(\varepsilon(X)u,\, v\big) + \big(u,\, \varepsilon(X)v\big) = 0\,.$$

Since, by definition $\chi: \mathfrak{k} \to \mathrm{End}(L)$ is the composite $\mathfrak{k} \xrightarrow{\tilde{\alpha}} \mathfrak{p}_2 \to \mathrm{End}(L)$ it now follows that for any $Y \in \mathfrak{k}$ and $u, v \in L$ one has

$$(4.3)\qquad \big(\chi(Y)u,\, v\big) + \big(u,\, \chi(Y)v\big) = 0\,.$$

Thus, the representation τ of K_1 on L is unitary. We now fix a hermitian metric (,) in V such that the representation τ of K_1 on V is unitary. Thus, we have for $Y \in \mathfrak{k}$, and $w,\ w' \in V$

$$(4.4)\qquad \big(\tau(Y)w,\, w'\big) + \big(w,\, \tau(Y)w'\big) = 0\,.$$

The inner products in L and V give rise to an inner product in $L \otimes V$ defined by

$$(4.5)\qquad \Big(\sum_i l_i \otimes v_i\,,\ \sum_j l'_j \otimes v'_j\Big) = \sum_{i,j} (l_i,\, l'_j)(v_i,\, v'_j)$$

for $l_i, l'_j \in L$ and $v_i, v'_j \in V$. With respect to this inner product the representation $\chi \otimes \tau$ of K_1 is unitary. Thus, each fibre in the bundle $E_{L\otimes V}$ obtains an inner product. Now, for $u, v \in C_c(E_{L\otimes V})$ we define

$$(4.6) \qquad (u, v) = \int_{G_1/K_1} (u(x), v(x))dx .$$

It is clear that the completion of $C_c(E_{L\otimes V})$ under the above inner product is the Hilbert space $L_2(E_{L\otimes V})$ of square integrable sections of the bundle $E_{L\otimes V}$.

We now define

$$(4.7) \qquad \begin{aligned} C(G_1, L \otimes V) &= L \otimes C^\infty(G_1) \otimes V \qquad \text{and} \\ L_2(G_1, L \otimes V) &= L \otimes L_2(G_1) \otimes V , \end{aligned}$$

where $L_2(G_1)$ is the space of square integrable complex valued functions on G_1. Similarly we define the spaces $C(G_1, L^\pm \otimes V)$ and $L_2(G_1, L^\pm \otimes V)$. We write $C^\pm(G_1, V)$ and $L_2^\pm(G_1, V)$ for $C(G_1, L^\pm \otimes V)$ and $L_2(G_1, L^\pm \otimes V)$ respectively. Also, we define as in §1,

$$(4.8) \qquad C(G_1, L \otimes V)^0 = \{u \in C(G_1, L \otimes V) \mid (\chi(k) \otimes r(k) \otimes \tau(k))(u) = u, \text{ for every } k \in K_1\} ,$$

and similarly $L_2(G_1, L \otimes V)^0$, $C^\pm(G_1, V)$, and $L_2^\pm(G_1, V)^0$. In $L_2(G_1, L \otimes V)$ we define an inner product by setting

$$(4.9) \qquad \left(\sum_i x_i \otimes f_i \otimes v_i , \sum_j x'_j \otimes f'_j \otimes v'_j\right) = \sum_{i,j} (x_i, x'_j)(f_i, f'_j)(v_i, v'_j) .$$

It is easy to see that as in the definition of the map $\eta\colon C(E_{L\otimes V}) \to C(G_1, L \otimes V)$ defined in §1, (see 1.14), every element $u \in L_2(E_{L\otimes V})$ gives rise to an element $\eta(u) \in L_2(G_1, L \otimes V)$ and thus defines an injection

$$(4.10) \qquad \eta\colon L_2(E_{L\otimes V}) \to L_2(G_1, L \otimes V) .$$

One has

$$(4.11) \qquad \eta(L_2(E_{L\otimes V})) = L_2(G_1, L \otimes V)^0 .$$

Moreover, it is easy to verify that η is an isometry. By this isometry η we often identify $L_2(E_{L\otimes V})$ with $L_2(G_1, L \otimes V)^0$.

Now consider the Dirac operator $D\colon C(E_{L\otimes V}) \to C(E_{L\otimes V})$ defined in §1 (see 1.12). We now have the following

LEMMA 4.2. *If $u, v \in C(E_{L\otimes V})$ and if u has compact support then*

$$(4.12) \qquad (u, Dv) = (Du, v) .$$

Proof. Identifying $C(E_{L\otimes V})$ with $C(G_1, L \otimes V)^0$ let $u = \sum_k l_k \otimes f_k \otimes w_k$ and $v = \sum_j l'_j \otimes f'_j \otimes w'_j$ where $l_k, l'_j \in L$, $f_k, f'_j \in C^\infty(G_1)$, and $w_k, w'_j \in V$. Note that f_k has compact support for every k. By Proposition 1.1, we have

$$Du = \sum_{i=1}^{2m} \sum_k \varepsilon(X_i)l_k \otimes \nu(X_i)f_k \otimes w_k ,$$

where $X_1, \cdots, X_{2m}$ is an orthonormal base for $\mathfrak{p}$. Thus, we have using (4.9),

$$\begin{aligned}(Du, v) &= \sum_{i=1}^{2m} \sum_{k,j} \big(\varepsilon(X_i)l_k, l'_j\big)\big(\nu(X_i)f_k, f'_j\big)(w_k, w'_j) \\ &= \sum_{i=1}^{2m} \sum_{k,j} \big(l_k, \varepsilon(X_i)l'_j\big)\big(f_k, \nu(X_i)f'_j\big)(w_k, w'_j)\end{aligned}$$

using the fact that $\big(\nu(X_i)f_k, f'_j\big) + \big(f_k, \nu(X_i)f'_j\big) = 0$ and (4.2). But, we have by Proposition 1.1,

$$Dv = \sum_{i=1}^{2m} \sum_j \varepsilon(X_i)l'_j \otimes \nu(X_i)f'_j \otimes w'_j .$$

So, using (4.9), we have

$$(u, Dv) = \sum_{i=1}^{2m} \sum_{k,j} \big(l_k, \varepsilon(X_i)l'_j\big)\big(f_k, \nu(X_i)f'_j\big)(w_k, w'_j) . \qquad \text{(q.e.d.)}$$

From (4.12) one easily deduces that if $u, v \in C(E_{L\otimes V})$ and if u has compact support, then

$$(u, \Box v) = (\Box u, v) = (Du, Dv) . \tag{4.13}$$

Recall that $\Box\big(C^{\pm}(E_V)\big) \subseteq C^{\pm}(E_V)$. The map η is an isometry of $C_c^{\pm}(E_V)$ with $C_c^{\pm}(G_1, V)^0$ and we can canonically identify the completion of $C_c^{\pm}(E_V)$ with $L_2^{\pm}(G_1, V)^0$.

We now define $\tilde{\Box}^+$ (resp. $\tilde{\Box}^-$) to be the weak realization of $\Box$ in $L_2^+(G_1, V)^0$ (resp. $L_2^-(G_1, V)^0$): i.e. the domain $D(\tilde{\Box}^+)$ of $\tilde{\Box}^+$ consists of all $\varphi \in L_2^+(G_1, V)^0$ for which there exists an element denoted $\Box\varphi \in L_2^+(G_1, V)^0$ such that

$$(\varphi, \Box\psi) = (\Box\varphi, \psi)$$

for every $\psi \in C_c^+(G_1, V)$ and then we define $\tilde{\Box}^+(\varphi) = \Box\varphi$. (Similarly for $\tilde{\Box}^-$.) We define $\tilde{D}^+$ and $\tilde{D}^-$ analogously.

We will have occasion to use the following lemma in §6.

LEMMA 4.3. *Let $\varphi \in C(E_{L\otimes V})$. Assume that $\|\varphi\| < \infty$ and $\|\Box\varphi\| < \infty$. Then $\|D\varphi\| < \infty$, $(\Box\varphi, \varphi) = (D\varphi, D\varphi)$. Moreover if $\Box\varphi = 0$ then $D\varphi = 0$.*

Proof. The lemma is a consequence of the completeness of the metric in G/K and can be proved with technique similar to the one in [1].

5. Some results of Harish-Chandra

Let $\mathcal{E}_{K_1}$ be the set of equivalence classes of irreducible representations of K_1. If π is a unitary representation of G_1 or K_1 we denote by $[\pi]$ the equivalence class which contains π. For any unitary representation σ of K_1 and $\delta \in \mathcal{E}_{K_1}$, we denote by $(\sigma : \delta)$ the multiplicity with which δ occurs in σ. We write $\big([\sigma] : \delta\big) = (\sigma : \delta)$. For any unitary representation π of G_1 let $\pi | K_1$ denote the restriction of π to K_1. Then $(\pi | K_1 : \delta)$ $(\delta \in \mathcal{E}_{K_1})$ depends only on the equivalence class $[\pi]$. We also write $\big([\pi] | K_1 : \delta\big)$ instead of $\big([\pi | K_1] : \delta\big)$. For any unitary representation π of G_1 on the representation space H, let H_d be the smallest closed invariant subspace of H which contains every irreducible closed invariant subspace of H. We denote by π_d the restriction of π to H_d. We call π_d (resp. H_d) the discrete part of π (resp. H). For any unitary representation σ and a finite dimensional representation τ of K_1, we put

$$([\sigma]:[\tau]) = \sum_{\delta \in \mathcal{E}_{K_1}} ([\sigma]:\delta)([\tau]:\delta) .$$

For any unitary representation π of G_1 or K_1, let π^* be the representation contragredient to π. It is obvious that $[\pi^*]$ depends only on $[\pi]$ and so we write $[\pi]^*$ instead of $[\pi^*]$.

We fix a $\lambda \in \mathcal{F}_0'$. We write τ and V for $\tau_{\lambda+\rho_n}$ and $V_{\lambda+\rho_n}$, where $\tau_{\lambda+\rho_n}$ is the irreducible representation of K_1 on $V_{\lambda+\rho_n}$ with highest weight $\lambda + \rho_n$. Now, for any $g \in G_1$, we denote by $\hat{\pi}^+(g)$ (resp. $\hat{\pi}^-(g)$) the canonical action of g on $L_2^+(E_V)$ (resp. $L_2^-(E_V)$) where $L_2^+(E_V)$ (resp. $L_2^-(E_V)$) is the Hilbert space of square integrable sections of the bundle $E_{L^+\otimes V}$ (resp. $E_{L^-\otimes V}$). Then $\hat{\pi}^+$ (resp. $\hat{\pi}^-$) is the unitary representation of G_1 which is induced by the representation $\chi^+ \otimes \tau$ (resp. $\chi^- \otimes \tau$) of K_1. Let $\mathcal{E}(G_1)$ be the set of equivalence classes of irreducible unitary representation of G_1. We call $\omega \in \mathcal{E}(G_1)$ a discrete class if ω contains a representation equivalent to the right (or equivalent to the left) regular representation restricted to a closed invariant subspace of $L_2(G_1)$. We denote by $\mathcal{E}_d(G_1)$ the set of discrete classes in $\mathcal{E}(G_1)$. $\mathcal{E}_d(G_1)$ is called the discrete series for G_1. $\mathcal{E}_d(G)$ is defined similarly. We have the obvious inclusion $\mathcal{E}_d(G) \subseteqq \mathcal{E}_d(G_1)$.

LEMMA 5.1. *We have*

$$[\tilde{\pi}^\pm] = \bigoplus_{\omega \in \mathcal{E}_d(G)} (\omega \,|\, K\colon [\chi^\pm \otimes \tau])\omega$$

where $\tilde{\pi}^\pm = (\hat{\pi}^\pm)_d$ *is the discrete part of the representation* $\hat{\pi}^\pm$ *of* G_1 *(also of* G*) on* $L_2^\pm(E_V)$*. The sum on the right hand side is finite.*

(The proof of this lemma is similar to that of Lemma 1.2 in [8].)

In the following we collect certain results of Harish-Chandra.

We denote by $\mathcal{C}(G_1)$ the Schwartz space of G_1 (for definition, see [4(c), §9]). For any $\omega \in \mathcal{E}_d(G_1)$, we denote by $d(\omega)$ the formal degree of ω (for definition see [4(a), §3]) and by Θ_ω the character of ω. Then Θ_ω is an invariant eigendistribution which is tempered. Hence Θ_ω is a locally summable function on G_1 which is analytic on G_1', where G_1' denotes the set of regular elements of G_1. We denote by E the orthogonal projection of $L_2(G_1)$ onto $L_2(G_1)_d$.

LEMMA 5.2. *For any finite dimensional representation σ of K_1, define*

$$\mathcal{E}_d(\sigma) = \{\omega \in \mathcal{E}_d(G_1) \,|\, (\omega \,|\, K_1\colon [\sigma]) \neq 0\} \,.$$

Then $\mathcal{E}_d(\sigma)$ *is a finite set.* (For a proof see [4(c), Lemma 70].)

LEMMA 5.3. *For any* $f \in \mathcal{C}(G_1)$ *put*

$$({}^0f)(x) = \sum\nolimits_{\omega \in \mathcal{E}_d(G_1)} d(\omega)\Theta_{\omega^*}(r(x)f) \qquad (x \in G_1) \,.$$

Then ${}^0f$ *is a continuous function on* G_1 *and* ${}^0f = Ef$.

For a proof see [4(a), Cor. 3 to Lemma 69 and a remark on p. 100].

LEMMA 5.4. *Let* φ *be a* K_1*-finite function in* $\mathcal{C}(G_1)$. *Then* ${}^0\varphi$ *is a* $\mathfrak{Z}$ *finite function which belongs to* $\mathcal{C}(G_1)$, *where* $\mathfrak{Z}$ *is the center of the universal enveloping algebra of* $\mathfrak{g}$. *Moreover, we have*

$$\Theta_\omega({}^0\varphi) = \Theta_\omega(\varphi) \,.$$

For a proof see Lemma 3.2 in [9]. Put

$$\mathfrak{h}' = \{H \in \mathfrak{h};\ \alpha(H) \neq 0\ ,\ \ (\alpha \in \Sigma)\} \,.$$

For any $\lambda \in \mathcal{F}'$ we denote by $\Theta_{\lambda+\rho}$ the unique tempered invariant eigendistribution on G_1 corresponding to $\lambda + \rho$ defined in [4(b), Th. 3]. Then we have

$$\Delta\,(\exp H)\cdot\Theta_{\lambda+\rho}\,(\exp H) = \sum\nolimits_{s \in W_G} \varepsilon(s) e^{s(\lambda+\rho)(H)} \qquad (H \in \mathfrak{h})$$

where $\Delta\,(\exp H) = \prod_{\alpha \in P}(e^{\alpha(H)/2} - e^{-\alpha(H)/2})$. (This is well defined on H and H_1.) For any $\lambda \in \mathcal{F}'$ we put

$$\varepsilon(\lambda + \rho) = \operatorname{sign}\left(\prod\nolimits_{\alpha \in P} \langle \lambda + \rho, \alpha \rangle\right) \,.$$

Then it is a well-known result due to Harish-Chandra that $\mathcal{E}_d(G) \subseteqq \mathcal{E}_d(G_1)$ is canonically parametrized by $\mathcal{F}'_0$ as follows. (We recall that the complexification G^C of G was assumed to be simply connected. See Remark 8.2 for the general case.)

THEOREM 5.5. *For any* $\lambda \in \mathcal{F}'_0$, *there exists a unique element* $\omega(\lambda + \rho) \in \mathcal{E}_d(G)$, *such that*

$$\Theta_{\omega(\lambda+\rho)} = (-1)^m \varepsilon(\lambda + \rho)\Theta_{\lambda+\rho}$$

where $m = (1/2)\dim G/K$. *Moreover, the mapping* $\mathcal{F}'_0 \to \mathcal{E}_d(G)$ *given by* $\lambda \mapsto \omega(\lambda + \rho)$ *is bijective.* (See [4(d), Theorem 16].)

6. The spaces of square integrable Dirac spinors

Let $H_2^+(E_V)$ (resp. $H_2^-(E_V)$) denote the subspace of $D(\tilde{\square}^+)$ (resp. $D(\tilde{\square}^-)$) consisting of all φ such that $\square^+(\varphi) = 0$ (resp. $\square^-(\varphi) = 0$). Since the operator $\tilde{\square}^\pm$ is elliptic one has $H_2^\pm(E_V) \subseteqq C^\pm(E_V)$. Also, using the fact that $\tilde{D}^\pm$ is elliptic and Lemma 4.3 it is easy to see that

$$H_2^\pm(E_V) = \{\varphi \in D(\tilde{D}^\pm) \,|\, D^\pm(\varphi) = 0\} \,.$$

The operators $\tilde{D}^{\pm}$, $\tilde{\Box}^{\pm}$ are closed operators and hence $H_2^{\pm}(E_V)$ is a closed subspace of $L_2^{\pm}(E_V)$. We also denote by $H_2^{\pm}(E_V)$ the subspaces of $L_2^{\pm}(G_1, V)^0$ which are images of the above subspaces under the isomorphism $\eta: L_2^{\pm}(E_V) \to L_2^{\pm}(G_1, V)^0$. We call these the spaces of square integrable Dirac spinors of type $\pm$ with coefficients in the bundle E_V.

We denote by $\pi^{\pm}$ the unitary representation of G on $H_2^{\pm}(E_V)$. Then we have the following

PROPOSITION 6.1. *Put*

$$\mathcal{E}_d(\lambda) = \{\omega \in \mathcal{E}_d(G) \,|\, \chi_\omega(\Omega) = \langle \lambda + 2\rho, \lambda \rangle\}$$

where χ_ω denotes the infinitesimal character of ω and Ω the Casimir operator. Then we have

$$[\pi^{\pm}] = \bigoplus_{\omega \in \mathcal{E}_d(\lambda)} (\omega | K: [\chi^{\pm} \otimes \tau])\omega$$

and the sum is finite.

Proof. By Proposition 3.2, $\pi^{\pm}(\Omega)$ is the scalar operator $\langle \lambda + 2\rho, \lambda \rangle$ where $\pi^{\pm}(\Omega)$ denotes the action of the Casimir operator Ω on $H_2^{\pm}(E_{V_{\lambda+\rho_n}})$. Using this fact, the assertion is proved just as Proposition 4.1 in [9].

Remark 6.1. In view of the above proposition $\pi^{\pm}$ is a finite sum of irreducible unitary representations and hence the character of $\pi^{\pm}$, denoted by Trace $\pi^{\pm}$, is defined as a distribution on G_1 (and also on G).

PROPOSITION 6.2. *The operator $\tilde{\Box}^{\pm}: D(\tilde{\Box}^{\pm}) \to L_2^{\pm}(E_V)$ has only finitely many eigenvalues and the discrete part $L_2^{\pm}(E_V)_d$ coincides with the sum of all eigenspaces of $\tilde{\Box}^{\pm}$.*

The proof of this proposition is similar to that of Proposition 5.1 in [9] in view of Proposition 3.2 and shall be omitted.

Remark 6.2. Let $\varphi \in D(\tilde{\Box}^{+})$ and assume that $\tilde{\Box}^{+}(\varphi) = \lambda\varphi$ where λ is a scalar. Then from the ellipticity of the operator $\Box$, it follows that $\varphi \in C^{+}(E_V)$. Now from Lemma 4.3, we conclude that $D\varphi \in L_2^{-}(E_V)$. Moreover, since D and $\Box = D^2$ commute, we have $\Box D\varphi = D\Box\varphi = \lambda \cdot D\varphi$. Then, by Proposition 6.2 it follows that $D\varphi \in L_2^{-}(E_V)_d$. Thus, $D\big(L_2^{+}(E_V)_d\big) \subseteqq L_2^{-}(E_V)_d$. Similarly, $D\big(L_2^{-}(E_V)_d\big) \subseteqq L_2^{+}(E_V)_d$.

PROPOSITION 6.3. *One has an exact sequence*

$$0 \longrightarrow H_2^{+}(E_V) \xrightarrow{\ i\ } L_2^{+}(E_V)_d \xrightarrow{\ D\ } L_2^{-}(E_V)_d \xrightarrow{\ j\ } H_2^{-}(E_V) \longrightarrow 0\ ,$$

where $H_2^{\pm}(E_V)$ is the zero eigenspace of $\Box$ in $L_2^{\pm}(E_V)_d$, i the inclusion and j the orthogonal projection onto $H_2^{-}(E_V)_d$.

Proof. If $\varphi \in H_2^+(E_V)$, then as noted in the beginning of §6, $D\varphi = 0$. Conversely, if $\varphi \in L_2^+(E_V)_d$ and $D\varphi = 0$, then $\square\varphi = D^2\varphi = 0$ and hence $\varphi \in H_2^+(E_V)$. Note that the kernel of j consists precisely of the sum of the nonzero eigenspaces of the operator $\square$ in $L_2^-(E_V)_d$. Now, suppose $\varphi \in L_2^-(E_V)_d$ and $\square\varphi = \lambda\varphi$, where $\lambda \neq 0$. Then $\varphi = D(\lambda^{-1}D\varphi)$ and $\lambda^{-1}D\varphi \in L_2^+(E_V)_d$. This implies that kernel of $j \subseteqq$ image of D. Finally, it is clear that image of $D \subseteqq$ kernel of j from the argument of Remark 6.2. (q.e.d.)

7. The difference formula

Now consider the representation $T^\pm$ of G_1 on $L_2^\pm(G_1, V)$ defined by $T_g^\pm = 1 \otimes l(g) \otimes 1$ where l denotes the left regular representation of G_1 on $L_2(G_1)$. Fix any K_1 finite function $\varphi \in C_c^\infty(G_1)$. Define the operator $T_\varphi^\pm$ on $L_2^\pm(G_1, V)$ by

$$T_\varphi^\pm = \int_{G_1} \varphi(g) T_g^\pm \, dg \; . \tag{7.1}$$

Note that $T_g^\pm$ and hence $T_\varphi^\pm$ leave $L_2^\pm(G_1, V)^0$ and $L_2^\pm(G_1, V)_d$ invariant and the canonical action $\tilde{\pi}^\pm(g)$ of $g \in G_1$ on $L_2^\pm(E_V)_d$ goes over by means of the identification map η into the restriction of $T_g^\pm$ on $L_2^\pm(G_1, V)_d^0$. Moreover, where

$$\tilde{\pi}_\varphi^\pm = \int_G \varphi(g)\tilde{\pi}^\pm(g) dg \; , \tag{7.2}$$

it is easy to see that under the identification map η the action of $\tilde{\pi}_\varphi^\pm$ on $L_2^\pm(E_V)_d$ goes over into the restriction of $T_\varphi^\pm$ on $L_2^\pm(G_1, V)_d^0$. Henceforth, wherever convenient we make these identifications without further mention. We remark that the orthogonal projection $L_2^\pm(E_V) \to L_2^\pm(E_V)_d$ is the restriction of $1 \otimes E \otimes 1$ where E is the orthogonal projection of $L_2(G_1)$ onto $L_2(G_1)_d$. $1 \otimes E \otimes 1$ itself is the orthogonal projection of $L_2^\pm(G_1, V)$ onto $L_2^\pm(G_1, V)_d$. Put

$$E_0^\pm = \int_K \chi^\pm(k) \otimes r(k) \otimes \tau(k) dk. \tag{7.3}$$

Then $E_0^\pm$ gives the orthogonal projection $L_2^\pm(G_1, V) \to L_2^\pm(G_1, V)^0$. It is clear that $L_2^\pm(G_1, V)_d$ is stable under $E_0^\pm$ and that the restriction of $E_0^\pm$ to $L_2^\pm(G_1, V)_d$ gives the orthogonal projection onto $L_2^\pm(G_1, V)_d^0$. Thus $E_0^\pm \circ (1 \otimes E \otimes 1)$ is the orthogonal projection of $L_2^\pm(G_1, V)$ onto $L_2^\pm(G_1, V)_d^0$.

PROPOSITION 7.1. *$\tilde{\pi}_\varphi^\pm$ is an operator of finite rank from $L_2^\pm(E_V)_d$ into itself. $T_\varphi^\pm \circ E_0^\pm \circ (1 \otimes E \otimes 1)$ is an integral operator of finite rank with an* $\mathrm{End}(L^\pm \otimes V)$ *valued C^∞ kernel function $K_\varphi^\pm$ which is given by*

$$K_\varphi^\pm(x, y) = \int_{K_1} {}^0\varphi(xky^{-1})(\chi^\pm \otimes \tau)(k) dk \; , \qquad \textit{for } (x, y) \in G_1 \times G_1 \; .$$

Moreover $\int_{G_1}$ Trace $K_\varphi^{\pm}(x, x)dx$ *exists and coincides with the trace of the operator* $\tilde{\pi}_\varphi^{\pm}$.

Proof. For a proof of this proposition see the proof of an exactly similar proposition, viz, Proposition 6.1 in [9].

We now have the following proposition, which is the analogue of the 'alternating sum formula' in [9].

PROPOSITION 7.2. *Let* p_λ *be the number of noncompact positive roots* α *such that* $\langle\lambda + \rho, \alpha\rangle > 0$. *Then we have*

$$\text{Trace}\, \tilde{\pi}_\varphi^{+} - \text{Trace}\, \tilde{\pi}_\varphi^{-} = (-1)^{p_\lambda}\Theta_{\omega(\lambda+\rho)}(\varphi) \, .$$

(For the definition of the right hand side see Theorem 5.5.)

Proof. By Proposition 1, we have

$$\text{Trace}\, \tilde{\pi}_\varphi^{\pm} = \int_{G_1} \text{Trace}\, K_\varphi^{\pm}(x, x)dx \, .$$

We put

$$A = \text{Trace}\, \tilde{\pi}_\varphi^{+} - \text{Trace}\, \tilde{\pi}_\varphi^{-} \, .$$

Then by Proposition 7.1, we have

$$A = \int_{G_1} dx \int_{K_1} \{\text{Trace}\, \chi^+(k) - \text{Trace}\, \chi^-(k)\} \cdot {}^0\varphi(xkx^{-1})\psi(x)dk$$

where ψ is the character of the irreducible representation $\tau(=\tau_{\lambda+\rho_n})$ of K_1. We now use Weyl's integral formula: for $f \in C^\infty(K_1)$,

$$\int_{K_1} f(k)dk = \frac{1}{[W_G]} \int_{H_1} |\Delta_k(h)|^2 \, dh \int_{K_1} f(khk^{-1})dk$$

where

$$\Delta_k(\exp X) = \prod\nolimits_{\alpha \in P_k} (e^{\alpha(X)/2} - e^{-\alpha(X)/2}) \qquad (X \in \mathfrak{h})$$

(as is well known $|\Delta_k(h)|^2$ is well defined on H and hence on H_1) and $[W_G]$ is the order of the group W_G. Note that Trace (χ^+) − Trace (χ^-) is $\mathrm{Ad}K_1$ invariant. Thus, we obtain

$$A = \frac{1}{[W_G]} \int_{G_1} dx \int_{H_1 \times K_1} \{\text{Trace}\, \chi^+(h) - \text{Trace}\, \chi^-(h)\} \, {}^0\varphi(xkhk^{-1}x^{-1})\psi(h) \, |\Delta_k(h)|^2 \, dh \, dk. \tag{7.4}$$

By Remark 2.3, we have

$$\text{Trace}\, \chi^+(h) - \text{Trace}\, \chi^-(h) = \Delta_n(h)$$

where

$$\Delta_n(\exp X) = \prod_{\alpha \in P_n} (e^{\alpha(X)/2} - e^{-\alpha(X)/2}) \qquad (X \in \mathfrak{h}) .$$

Also, remembering that τ is an irreducible representation of K_1 with highest weight $\lambda + \rho_n$, we have using Weyl's character formula

$$\psi(\exp X) = \sum_{s \in W_G} \frac{\varepsilon(s) e^{s(\lambda+\rho)(X)}}{\Delta_k(\exp X)} .$$

Also, using Theorem 5.5 and observing that $(-1)^m \varepsilon(\lambda + \rho) = (-1)^{m-p_\lambda}$ we have

$$\Theta_{\omega(\lambda+\rho)}(\exp X) = (-1)^{m-p_\lambda} \sum_{s \in W_G} \frac{\varepsilon(s) e^{s(\lambda+\rho)(X)}}{\Delta(\exp X)}$$

for $\exp X \in H_1'$ $(= \exp \mathfrak{h}')$. For $h \in H_1'$, define

$$\Phi_{\omega(\lambda+\rho)}(h) = \Delta(h) \Theta_{\omega(\lambda+\rho)}(h) .$$

One knows that the function $\Phi_{\omega(\lambda+\rho)}$ can be extended to a C^∞ function on H_1. (See 4(c), Lemma 31.) Finally, noting that

$$\overline{\Delta_k(h)} = (-1)^q \Delta_k(h)$$

where $q = (1/2) \dim K/H$ and making the substitutions in 7.4, we get

$$A = \frac{(-1)^{p_\lambda}}{W_G} \int_{G_1} dx \int_{H_1 \times K_1} (-1)^{m+q} \Phi_{\omega(\lambda+\rho)}(h) \Delta(h) {}^0\varphi(xkhk^{-1}x^{-1}) dh\, dk .$$

Now, for $h \in H_1'$ put

$$F_{{}^0\varphi}(h) = \Delta(h) \int_G {}^0\varphi(xhx^{-1}) dx .$$

We remark that the restriction of $F_{{}^0\varphi}$ to each connected component of H_1' extends as a continuous function to its closure. (See [4(d)], Lemma 27 and its proof.) Then arguing as in the proof of Prop. 6.2 in [9], we see that

$$A = (-1)^{p_\lambda + m + q} \frac{1}{[W_G]} \int_{H_1} \Phi_{\omega(\lambda+\rho)}(h) F_{{}^0\varphi}(h) dh .$$

By Lemma 5.4, ${}^0\varphi$ is a $\mathfrak{Z}$ finite function and it follows from [4(d), Lemma 79] that

$$\Theta_{\omega(\lambda+\rho)}({}^0\varphi) = (-1)^{m+q} \frac{1}{[W_G]} \int_{H_1} F_{{}^0\varphi}(h) \Phi_{\omega(\lambda+\rho)}(h) dh .$$

Thus, we have

$$A = (-1)^{p_\lambda} \Theta_{\omega(\lambda+\rho)}({}^0\varphi) .$$

But, we have from Lemma 5.4

$$\Theta_{\omega(\lambda+\rho)}({}^0\varphi) = \Theta_{\omega(\lambda+\rho)}(\varphi) .$$

This proves Proposition 7.2. (q.e.d.)

Now observe that $\tilde{\pi}_\varphi^+ | H_2^+(E_V)$, $\tilde{\pi}_\varphi^+$, $\tilde{\pi}_\varphi^-$, and $\tilde{\pi}_\varphi^- | H_2^-(E_V)$ give rise to an endomorphism of finite rank (see Proposition 7.1) of the exact sequence of Proposition 6.3. It is a simple consequence of the exactness of that sequence that the alternating sum of the traces of these maps is zero, i.e.,

$$\text{Trace}\,(\tilde{\pi}_\varphi^+ | H_2^+(E_V)) - \text{Trace}\,\tilde{\pi}_\varphi^+ + \text{Trace}\,\tilde{\pi}_\varphi^- - \text{Trace}\,(\tilde{\pi}_\varphi^- | H_2^-(E_V)) = 0\ .$$

Hence,

$$\text{Trace}\,(\tilde{\pi}_\varphi^+ | H_2^+(E_V)) - \text{Trace}\,(\tilde{\pi}_\varphi^- | H_2^-(E_V)) = \text{Trace}\,\tilde{\pi}_\varphi^+ - \text{Trace}\,\tilde{\pi}_\varphi^-\ .$$

The following Theorem now follows from Proposition 7.2.

THEOREM 1. *Let* $\lambda \in \mathcal{F}_0'$ *be given and let* $\tau_{\lambda+\rho_n}$ *be the irreducible unitary representation of* K_1 *on the space* $V_{\lambda+\rho_n}$, *with highest weight* $\lambda + \rho_n$. *Let* $E_{L^\pm \otimes V_{\lambda+\rho_n}}$ *be the vector bundle on* $G/K(= G_1/K_1)$ *induced by* $\chi^\pm \otimes \tau_{\lambda+\rho_n}$ *and let* $C^\pm(E_{V_{\lambda+\rho_n}})$ *be the space of* C^∞ *sections of the bundle* $E_{L^\pm \otimes V_{\lambda+\rho_n}}$. *Let* $\pi_\lambda^\pm$ *denote the unitary representations of* G *on* $H_2^\pm(E_{V_{\lambda+\rho_n}})$, *the space of square integrable Dirac spinors of type* $\pm$ *with coefficients in the bundle* $E_{V_{\lambda+\rho_n}}$. *Let* p_λ *be the number of noncompact positive roots* α *such that* $\langle \lambda + \rho, \alpha \rangle < 0$. *Then, we have*

$$\text{Trace}\,\pi_\lambda^+ - \text{Trace}\,\pi_\lambda^- = (-1)^{p_\lambda} \Theta_{\omega(\lambda+\rho)}\ ,$$

where Trace $\pi^\pm$ *denotes the character of* $\pi^\pm$ *and* $\Theta_{\omega(\lambda+\rho)}$ *the character of the discrete class* $\omega(\lambda + \rho)$ *determined by* $\lambda + \rho$. (See Theorem 5.5.)

8. The vanishing theorem

Recall the subset W^1 of the Weyl group $W(\mathfrak{h}^C, \mathfrak{g}^C)$. (See 2.5.) For $\sigma \in W^1$, we define

$$j(\sigma) = \begin{cases} + & \text{if } \varepsilon(\sigma) = +1\ , \\ - & \text{if } \varepsilon(\sigma) = -1\ . \end{cases} \tag{8.1}$$

LEMMA 8.1. *Let* ν *be an irreducible representation of* $\mathfrak{g}^C$ *in a finite dimensional complex vector space* F_λ *with highest weight* λ. *Let*

$$L \otimes F_\lambda = V_{\xi_1} \oplus V_{\xi_2} \oplus \cdots V_{\xi_i} \tag{8.2}$$

be a decomposition of the $\mathfrak{k}^C$ *module* $L \otimes F_\lambda$ *into a direct sum of irreducible* $\mathfrak{k}^C$ *modules* V_{ξ_l}, *where* ξ_l *is the highest weight of the representation of* $\mathfrak{k}^C$ *in* V_{ξ_l}. *Then we have*

$$|\lambda + \rho| \geqq |\xi_l + \rho_k| \tag{8.3}$$

and the equality

$$|\lambda + \rho| = |\xi_l + \rho_k| \tag{8.4}$$

holds if and only if there exists $\sigma \in W^1$ *such that*

$$\xi_l = \sigma(\lambda + \rho) - \rho_k \; . \tag{8.5}$$

The mapping $\sigma \mapsto \xi_\sigma = \sigma(\lambda + \rho) - \rho_k$ *is a bijection of* W^1 *onto the set of highest weights* ξ_l *appearing in the decomposition* (8.2) *and satisfying the equality* (8.4).

As a weight of $\nu_\lambda \otimes \chi$, ξ_σ *occurs with multiplicity one and a weight vector of* ξ_σ *is, up to a scalar multiple, of the form* $e_{\sigma\lambda} \otimes e_{\sigma\rho-\rho_k}$ *where* $e_{\sigma\lambda}$ *is a weight vector in* F_λ, *belonging to the weight* $\sigma\lambda$ *and* $e_{\sigma\rho-\rho_k}$ *a weight vector in* L *belonging to the weight* $\sigma\rho - \rho_k$.

Proof. The proof of this lemma is similar to the proof of [8, Lemma 5.12]. We just indicate a few points. We have $\xi_l = \mu + f$, where μ is a weight of ν_λ and f a weight of χ. Thus, $\xi_l = \mu + \rho_n - \langle\Phi\rangle$, for some $\Phi \subseteqq P_n$ (see Remark 2.1). Hence, $\xi_l + \rho_k = \mu + \rho - \langle\Phi\rangle$. Now, the fact that $|\lambda + \rho| \geqq |\xi_l + \rho_k|$ and that equality holds if and only if ξ_l has the form (8.5), follows using [8, Lemma 5.8] and observing that $\rho - \langle\Phi\rangle$ is a weight of the irreducible representation ν_ρ of $\mathfrak{g}^C$ with highest weight ρ. (See [8, Lemma 5.9].) The proof can now be completed as in the cited reference. (q.e.d.)

Remark 8.1. Let D be the set of linear forms λ on $\mathfrak{h}^C$ such that $2\langle\lambda, \alpha\rangle/\langle\alpha, \alpha\rangle$ is a nonnegative integer for every $\alpha \in P$. Then one can easily see that the map

$$D \times W^1 \longrightarrow \mathcal{F}_0'$$

given by

$$(\lambda, \sigma) \longmapsto \lambda^{(\sigma)} \; ,$$

where $\lambda^{(\sigma)} = \sigma(\lambda + \rho) - \rho$, is a bijection (see [8, Lemma 6.4]).

We now have the following

THEOREM 2. *Let* $\lambda \in D$ *and let* $\lambda^{(\sigma)} = \sigma(\lambda + \rho) - \rho$ *for every* $\sigma \in W^1$, so *that* $\lambda^{(\sigma)} \in \mathcal{F}_0'$ *(see Remark* 8.1). *Now let* $\tau_{\lambda^{(\sigma)}+\rho_n}$ *be the irreducible unitary representation of* K_1 *on the space* $V_{\lambda^{(\sigma)}+\rho_n}$ *with highest weight* $\lambda^{(\sigma)} + \rho_n$. *Let* $E_{L^\pm \otimes V_{\lambda^{(\sigma)}+\rho_n}}$ *be the vector bundle on* G/K, *induced by* $\chi^\pm \otimes \tau_{\lambda^{(\sigma)}+\rho_n}$. *Let* $H_2^j(E_{V_{\lambda^{(\sigma)}+\rho_n}})$ *denote the space of square integrable Dirac spinors of type* j $(j = +$ *or* $-)$ *with coefficients in* $E_{V_{\lambda^{(\sigma)}+\rho_n}}$. *Now assume that*

$$\langle\sigma\lambda, \alpha\rangle \neq 0 \quad \textit{for any noncompact root } \alpha. \tag{8.6}$$

Then $H_2^j(E_{V_{\lambda^{(\sigma)}+\rho_n}}) = 0$, *if* $j \neq j(\sigma)$, *where* $j(\sigma)$ *is defined by* (8.1).

Proof. Assume $j \neq j(\sigma)$. Since $\lambda^{(\sigma)} + \rho_n = \sigma\lambda + \sigma\rho - \rho_k$, we have

(8.7) $$V_{\lambda^{(\sigma)}+\rho_n} \otimes L^j \subseteqq V_{\sigma\lambda} \otimes V_{\sigma\rho-\rho_k} \otimes L^j .$$

Here consider $V_{\sigma\lambda} \otimes L^j$. Let

(8.8) $$V_{\sigma\lambda} \otimes L^j = \sum V_{\xi_l}$$

be a direct sum decomposition of $V_{\sigma\lambda} \otimes L^j$ into irreducible $\mathfrak{k}^C$ modules V_{ξ_l}, where ξ_l is the highest weight of the representation of $\mathfrak{k}^C$ in V_{ξ_1}. In view of Lemma 8.1, one sees that for each ξ_l

(8.9) $$|\lambda + \rho| \geqq |\xi_l + \rho_k| .$$

We assert that equality cannot occur when the condition (8.6) is satisfied. For, we have by Lemma 2.2,

(8.10) $$L^j = \sum_{\tau \in W^1, j(\tau)=j} V_{\tau\rho-\rho_k} .$$

Thus, if for some τ appearing on the right hand side of (8.10), $V_{\xi_l} \subseteqq V_{\sigma\lambda} \otimes V_{\tau\rho-\rho_k}$, for some ξ_l such that $|\xi_l + \rho_k| = |\lambda + \rho|$, then we will have

(8.11) $$\sigma\lambda = \tau\lambda ,$$

for otherwise no weight vector of the form $e_{\sigma'\lambda} \otimes e_{\sigma'\rho-\rho_k}$, which is given by Lemma 8.1 can belong to $V_{\sigma\lambda} \otimes V_{\tau\rho-\rho_k}$. We will now prove the assertion that equality does not hold in (8.9), by showing that (8.11) is impossible. For this first note that $\sigma \neq \tau$ since $j(\tau) = j \neq j(\sigma)$. Hence also $\sigma P \neq \tau P$. But since σ and τ belong to W^1, it follows from the definition of W^1, that P_k is contained in σP as well as τP. Thus, there exists a noncompact $\alpha \in \sigma P$ such that $-\alpha \in \tau P$. Since $\lambda \in D$, $\langle \lambda, \alpha' \rangle \geqq 0$, for every $\alpha' \in P$ and hence we conclude that

(8.12) $$\langle \sigma\lambda, \alpha \rangle \geqq 0 \quad \text{and} \quad \langle \tau\lambda, -\alpha \rangle \geqq 0 .$$

But it follows from the assumption (8.6) of the theorem that $\langle \sigma\lambda, \alpha \rangle > 0$. Using (8.12) we then conclude that $\sigma\lambda \neq \tau\lambda$. But this contradicts (8.11). Thus when the condition (8.6) of the theorem is satisfied we have concluded for every ξ_l such that V_{ξ_l} occurs in the decomposition (8.8) that

(8.13) $$|\lambda + \rho| > |\xi_l + \rho_k| .$$

Now, by (8.7) and (8.8), we have

(8.14) $$V_{\lambda^{(\sigma)}+\rho_n} \otimes L^j \subseteqq \sum V_{\sigma\rho-\rho_k} \otimes V_{\xi_l} \subseteqq \sum L \otimes V_{\xi_l} .$$

Suppose now $H_2^j(E_{V_{\lambda^{(\sigma)}+\rho_n}}) \neq 0$. Then by Proposition 3.1 there exists a nonzero square integrable eigenvector in $C^j(E_{V_{\lambda^{(\sigma)}+\rho_n}})$ belonging to the eigenvalue $\langle \lambda^{(\sigma)} + 2\rho, \lambda^{(\sigma)} \rangle$ for the action of Ω on $C^j(E_{V_{\lambda^{(\sigma)}+\rho_n}})$. Note that $\langle \lambda^{(\sigma)} + 2\rho, \lambda^{(\sigma)} \rangle = \langle \lambda + 2\rho, \lambda \rangle$. Now (8.14) implies that for some V_{ξ_l} occuring in the decomposition (8.8), with the obvious notation, there exists a nonzero square integrable eigenvector in $C(E_{L\otimes V_{\xi_l}})$ belonging to the eigenvalue $\langle\lambda + 2\rho, \lambda\rangle$ for the action of Ω on $C(E_{L\otimes V_{\xi_l}})$. Let ψ be this eigenvector. Now consider the Dirac operator D and the operator $\square = D^2$ on $C(E_{L\otimes V_{\xi_l}})$. Denoting by π the action of $U(\mathfrak{g}^C)$ on $C(E_{L\otimes V_{\xi_l}})$ we have by Proposition 3.1,

$$\square\psi = -\pi(\Omega)\psi + \langle\xi_l - \rho_n + 2\rho, \xi_l - \rho_n\rangle\psi ,$$

i.e.

$$\square\psi = -\langle\lambda + 2\rho, \lambda\rangle\psi + \{\langle\xi_l + \rho_k, \xi_l + \rho_k\rangle - \langle\rho, \rho\rangle\}\psi ,$$

i.e.

$$\square\psi = \{|\xi_l + \rho_k|^2 - |\lambda + \rho|^2\}\psi .$$

In particular $\square\psi$ and hence, by Lemma 4.3, $D\psi$ also are square integrable. Thus, by Lemma 4.3 again,

$$(D\psi, D\psi) = (\square\psi, \psi) = \{\langle\xi_l + \rho_k, \xi_l + \rho_k\rangle - \langle\lambda + \rho, \lambda + \rho\rangle\}(\psi, \psi) .$$

But the left hand side is nonnegative while the right hand side is strictly negative in view of (8.13). This is a contradiction.

Thus, we conclude that $H_2^j(E_{V_{\lambda(\sigma)+\rho_n}}) = 0$. (q.e.d.)

Now, let the notation be as in Theorem 2 and assume that the condition (8.6) is satisfied. Then, from Theorem 2 we have

$$H_2^j(E_{V_{\lambda(\sigma)+\rho_n}}) = 0 \quad \text{for } j \neq j(\sigma) .$$

It follows that

$$\text{Trace}\,\pi^+_{\lambda(\sigma)} - \text{Trace}\,\pi^-_{\lambda(\sigma)} = \varepsilon\cdot\text{Trace}\,\pi^{j(\sigma)}_{\lambda(\sigma)} ,$$

where $\text{Trace}\,\pi^\pm_{\lambda(\sigma)}$ is the character of the unitary representation $\pi^\pm_{\lambda(\sigma)}$ of G on $H_2^\pm(E_{V_{\lambda(\sigma)+\rho_n}})$ and where $\varepsilon = +1$ or -1 according as $j(\sigma) = +$ or $-$. It now follows from Theorem 1, §7, that

$$\text{Trace}\,\pi^{j(\sigma)}_{\lambda(\sigma)} = \Theta_{\omega(\lambda(\sigma)+\rho)} .$$

But, by Proposition 6.1, $\pi^{j(\sigma)}_{\lambda(\sigma)}$ is a finite sum of irreducible unitary representations of G. Using [4(a), Theorem 6 and its corollary] we obtain the following

THEOREM 3. *Let the notation be as in Theorem 2 and let $\pi^\pm_{\lambda(\sigma)}$ denote the unitary representation of G on $H_2^\pm(E_{V_{\lambda(\sigma)+\rho_n}})$. Assume that the condition* (8.6) *of Theorem 2 is satisfied. Then we have*

$$[\pi^{j(\sigma)}_{\lambda(\sigma)}] = \omega(\lambda^{(\sigma)} + \rho) ,$$

where $[\pi^{j(\sigma)}_{\lambda(\sigma)}]$ denotes the equivalence class of the representation $\pi^{j(\sigma)}_{\lambda(\sigma)}$ and $\omega(\lambda^{(\sigma)} + \rho)$ is the discrete class corresponding to $\lambda^{(\sigma)} \in \mathcal{F}'_0$. (See Theorem 5.5.)

Remark 8.2. When the complexification G^C of G is not assumed to be simply connected, the set $\mathcal{E}_d(G)$ of discrete classes of G is parametrized by the subset $\mathcal{F}_0'(G)$ of $\mathcal{F}_0'$ defined as follows:

$$\mathcal{F}_0'(G) = \{\lambda \in \mathcal{F}_0' \mid \lambda \text{ gives rise to a character of the compact Cartan subgroup of } G \text{ with Lie algebra } \mathfrak{h}.\}.$$

The correspondence between $\mathcal{E}_d(G)$ and $\mathcal{F}_0'(G)$, due to Harish-Chandra, is again in the sense of Theorem 5.5. Now, given $\lambda \in \mathcal{F}_0'(G)$, the representation $\chi \otimes \tau_{\lambda+\rho_n}$ of $\mathfrak{k}$ gives rise to a representation, also denoted $\chi \otimes \tau_{\lambda+\rho_n}$, of $K \subset G$, the subgroup with Lie algebra $\mathfrak{k}$. Thus G acts on the vector bundle $E_{L\otimes V_{\lambda+\rho_n}}$ on G/K, on the space of sections and also on the spaces $H_2^{\pm}(E_{V_{\lambda+\rho_n}})$. In particular, when the condition (8.6) is satisfied, the discrete class $\omega(\lambda + \rho) \in \mathcal{E}_d(G)$ is realized by Theorem 3.

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