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On a Construction of Representations and a Problem of Enright

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§ 1. Introduction

This paper is an attempt to understand the concept of completions of certain (infinite dimensional) representations of complex semi simple Lie algebras as introduced by T. Enright in [3, 4]. The completion is a process (a functor C_α as we will see later on) of obtaining new representations from a given one, the latter sitting as a sub-representation. One of the standard techniques in representation theory of building new representations is to induce from representations of sub-groups. This is very extensively used in Harish Chandra's Theory of representations [5]. The process of completion is equally effective as demonstrated by Enright; he is able to build Harish Chandra modules for real semi simple Lie algebras and give an algebraic Theory of the discrete series and more generally fundamental series representations. However, his construction of completions is complicated and has to be done in two stages; first for sl_2 and then for general Lie algebras. Also the description of structure of the completions is somewhat indirect.

In an attempt to remove these drawbacks, we have come across a certain functor D_α (see § 2). There are indications (e.g. Proposition 3.4) to show that this functor may turn out to be an important one. However, the consequences are not well understood as yet. The completion functor C_α sits as a naturally defined 'sub functor' of D_α. This description is quite transparent and as an illustration of this fact, we solve an important problem posed by Enright [3, § 4]. We remark here that our construction makes sense in the setup of infinite dimensional Lie algebras introduced by Kăc and Moody (cf. [6] for an exposition of these Lie algebras; see also the remark at the end of § 3 here).

The paper is arranged as follows: In § 2, we give the construction of functors D_α and C_α and prove some elementary properties. In § 3, we investigate what happens on tensoring with finite dimensional representations. In § 4, we consider the problem posed by Enright.

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§ 2. Constructions for functors D_α and C_α

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra; $\mathfrak{b} \supseteq \mathfrak{h}$ be respectively a Borel subalgebra and a Cartan subalgebra. Let Φ be the root system of $(\mathfrak{g}, \mathfrak{h})$; we have,

A. Bouaziz has independently given a proof of Theorem 4.2 above; this is contained in his paper (unpublished) "Sur les representations des Algebres de Lie semi-simples construites par T. Enright"

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$\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in\Phi}\mathfrak{g}^\alpha$. Let Φ^+ be the positive root system corresponding to $\mathfrak{b}$ and Δ the set of simple roots. Choose a Chevalley basis $\{x_\varphi, y_\varphi\}_{\varphi\in\Phi^+}\cup\{H_\alpha\}_{\alpha\in\Delta}$. Let $\mathfrak{n}^+$ (respectively $\mathfrak{n}^-$) be the nilpotent subalgebra corresponding to Φ^+ (respectively $-\Phi^+$). For a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, let $\mathrm{U}(\mathfrak{a})$ denote the universal envelopping algebra of $\mathfrak{a}$.

Let $\mathscr{A}$ be the category of $\mathfrak{g}$-modules A satisfying (i) A is $\mathfrak{h}$-semisimple (i.e. a weight module) and (ii) $\mathrm{U}(\mathfrak{n}^-)$-action on A is torsion-free. Let $\mathscr{I}$ be the subcategory of $\mathscr{A}$ consisting of modules M which satisfy a further condition (iii) M is $\mathrm{U}(\mathfrak{n}^+)$-finite (This is equivalent to saying that x_φ for $\varphi\in\Phi^+$ acts locally nilpotently on M.)

Fix a simple root $\alpha\in\Delta$. For brevity, we write $y_\alpha=y$, $x_\alpha=x$ and $H_\alpha=H$. We define a functor $D_\alpha\colon\mathscr{A}\to\mathscr{A}$ as follows: Let $A\in\mathscr{A}$. Consider the set A' of formal symbols $\{y^{-n}a\,|\,0\leqq n\in\mathbf{Z}, a\in A\}$. Define an equivalence relation $\sim$ on A' by: $y^{-n}a\sim y^{-k}a'$ iff $y^k a=y^n a'$. Let $D_\alpha(A)=A'/\sim$. It is fairly easy to see that $D_\alpha(A)$ has a vector-space structure. We now give it a $\mathfrak{g}$-module structure. We need a lemma.

Lemma 2.1. *Given $z\in\mathfrak{g}$ and $0\leqq r\in\mathbf{Z}$, $\exists\, 0\leqq s\in\mathbf{Z}$ such that $y^s z=u\cdot y^r$ for some $u\in\mathrm{U}(\mathfrak{g})$.*

The proof of this lemma is clear from the identity:

$$y^t\cdot z=\sum_{j=0}^{t}\binom{t}{j}(ad\,y)^j(z)\cdot y^{t-j}$$

and the fact that $ad\,y$ is nilpotent on $\mathfrak{g}$. We note here that it is possible to write down u in terms of the Chevalley basis; we will need this precise information later (Proposition 2.5, 3.4).

Let $y^{-r}a\in D_\alpha(A)$ and $z\in\mathfrak{g}$ be given. Define $z\cdot y^{-r}a=y^{-s}u\,a\in D_\alpha(A)$ where s is the integer given by Lemma 2.1. We now have

Proposition 2.2. *The above action of $\mathfrak{g}$ on $D_\alpha(A)$ is well-defined and makes it a $\mathfrak{g}$-module.*

Proof. The proof is quite straight-forward and uses Lemma 2.1 again and again. We omit the details.

It is easy to see that $D_\alpha(A)$ is a weight-module; If $a\in A$ is of weight $\mu\in\mathfrak{h}^*$, then $y^{-n}a$ is of weight $\mu+n\alpha$. Further, it is readily checked that $D_\alpha(A)$ is $\mathrm{U}(\mathfrak{n}^-)$-torsion-free as A is so. Thus $D_\alpha(A)\in\mathscr{A}$ again. Also $A\hookrightarrow D_\alpha(A)$. Next, given $f\in\mathrm{Hom}_{\mathfrak{g}}(A_1, A_2)$ define $D_\alpha f\in\mathrm{Hom}_{\mathfrak{g}}(D_\alpha(A_1), D_\alpha(A_2))$ by $(D_\alpha(f))(y^{-n}a)=y^{-n}f(a)$. Then D_α is indeed a functor from $\mathscr{A}\to\mathscr{A}$. We call this the α-localization functor.

We now turn to the subcategory $\mathscr{I}$. Let $M\in\mathscr{I}$. Then $D_\alpha(M)$ is defined and is in $\mathscr{A}$. However, $D_\alpha(M)\notin\mathscr{I}$ as x_α does not act locally nilpotently on it. (We will see later that x_φ, $\varphi\in\Phi^+$, $\varphi\neq\alpha$ always acts locally nilpotently on $D_\alpha(M)$ for $M\in\mathscr{I}$.) Define

$$C_\alpha(M)=\{\xi\in D_\alpha(M)\,|\,x_\varphi^{a(\varphi)}\xi=0 \text{ for some } a(\varphi)\gg 0,\ \varphi\in\Phi^+ \text{ arbitrary}\}.$$

It is easy to check that $C_\alpha(M)$ is a $\mathfrak{g}$-submodule. (Observe that in Lemma 2.1, y can be replaced by any element z such that adz is nilpoint. In our case, we take $z=x_\varphi$ and use the fact that x_φ acts locally nilpotently on M.) Also, $M\hookrightarrow C_\alpha(M)$. It is quite clear that given M_1, $M_2\in\mathscr{I}$ and $f\in\mathrm{Hom}_{\mathfrak{g}}(M_1, M_2)$, $(D_\alpha(f))(C_\alpha(M_1))\subseteq C_\alpha(M_2)$. Thus $D_\alpha(f)/C_\alpha(M_1)\in\mathrm{Hom}_{\mathfrak{g}}(C_\alpha(M_1), C_\alpha(M_2))$. Thus C_α is a functor from $\mathscr{I}$ to $\mathscr{I}$.

We recall the definition of completions (cf. [3, §3]): A module $P \in \mathscr{I}$ is said to be complete with respect to α if for every integer $n \geqq 0$, $y^{n+1}: P[n]^x \to P[-n-2]^x$ is bijective; here, $P[n] = \{p \in P \mid H \cdot p = n \cdot p\}$ and $P[n]^x = \{p \in P[n] \mid x \cdot p = 0\}$. A module $M' \in \mathscr{I}$ is said to be a completion of a module $M \in \mathscr{I}$ if (i) M' is complete with respect to α, (ii) $M \hookrightarrow M'$ and (iii) M'/M is y-finite.

We now prove:

Theorem 2.3. *Let $M \in \mathscr{I}$ be any module. Then*

i) *$C_\alpha(M)$ is a completion of M.*

ii) *If M' is any completion of M then $C_\alpha(M) \overset{f}{\simeq} M'$ such that $f/M =$ identity.*

Proof. (i) By the very definition of $C_\alpha(M)$, $C_\alpha(M)/M$ is y-finite. So we have only to show that $C_\alpha(M)$ is complete with respect to α.

Let $0 \leqq k \in \mathbf{Z}$ and $0 \neq \xi \in C_\alpha(M)[-k-2]^x$. Since ξ is an x-invariant of weight <0 and $C_\alpha(M)/M$ is y-finite, it is clear that $\xi \in M$ (Observe that $C_\alpha(M)/M$ is a direct sum of finite dimensional $\{x, y, H\}$-modules). Consider $\hat{\xi} = y^{-k-1} \cdot \xi \in D_\alpha(M)$. We then have,

$$\begin{aligned} x \cdot y^{-k-1} \cdot \xi &= y^{-k-2}(xy - (k+2)(H+k+1))\xi \\ &= y^{-k-2}(H - (k+2)(H+k+1))\xi \qquad (\text{as } x \cdot \xi = 0) \\ &= y^{-k-2}(-k-2-(k+2)(-k-2+k+1))\xi \\ &= 0. \end{aligned}$$

Thus $x \cdot \hat{\xi} = 0$. As remarked earlier, x_φ for $\varphi \neq \alpha$ always acts locally nilpotently on the whole of $D_\alpha(M)$. (We will prove this later.) Thus $\hat{\xi} \in C_\alpha(M)$ by definition. Also, $\hat{\xi} \in C_\alpha(M)[k]^x$ and $y^{k+1}\hat{\xi} = \xi$. We have therefore proved that $C_\alpha(M)$ is complete.

(ii) Let M' be any completion of M. Consider the inclusion $M \overset{i}{\hookrightarrow} M'$. By functoriallity of C_α, we get a map $C_\alpha(M) \xrightarrow{C_\alpha(i)} C_\alpha(M')$. It is easy to see that $C_\alpha(i)$ is an injection. Also, $C_\alpha(M')/M$ is y-finite (as $C_\alpha(M')/M'$ and M'/M are y-finite) from which it is clearly seen that $C_\alpha(i)$ is surjective too. (Observe that both $C_\alpha(M)$ and $C_\alpha(M')$ can be considered to be submodules of $D_\alpha(M')$). It now remains to be proved that $C_\alpha(M') = M'$.

If $C_\alpha(M') \neq M'$, then choose a non-zero x-invariant θ in $C_\alpha(M')/M'$. As θ is of non-negative weight k, it can be easily seen that $\exists \xi \in C_\alpha(M')$ such that $x \cdot \xi = 0$ and $\bar{\xi} = \theta$. (More generally, in the category $\mathscr{I}$, a 'dominant' invariant can always be lifted.) From sl_2-theory, it follows immediately that $y^{k+1}\theta = 0$ i.e. $y^{k+1}\xi \in M'$ and moreover, $x \cdot y^{k+1}\xi = 0$. Now $y^{k+1}\xi \in M'[-k-2]^x$ and M' is complete so $\exists \xi_1 \in M'$ such that $y^{k+1}\xi_1 = y^{k+1}\xi$. Thus $\xi_1 = \xi \in M'$ which is a contradiction as $\bar{\xi} = \theta \neq 0$. This proves that $C_\alpha(M') = M'$.

This completes the proof of the theorem.

Corollary 2.4 (Enright, [3, Proposition 3.3]). *For $M \in \mathscr{I}$, a completion exists and any two completions are naturally isomorphic.*

Proof is immediate.

Remark. The main thrust of the above theorem is the fact that one gets a concrete model $C_\alpha(M)$ for the completion of M. The advantage of having such explicit description at ones disposal is quite obvious.

Proposition 2.5. *Let $M \in \mathscr{I}$, then x_φ for $(\alpha \neq)\varphi \in \Phi^+$ acts locally nilpotently on the whole of $D_\alpha(M)$.*

Proof. The proof essentially uses the fact that for $i \geqq 1$, j arbitrary $(\in \mathbf{Z})$, $i\varphi + j\alpha \in \Phi \Rightarrow i\varphi + j\alpha \in \Phi^+$. To simplity the computations, we assume that $i\varphi + j\alpha \notin \Phi$ $\forall\, i \geqq 2$. The argument in the general case is similar though slightly more complicated.

We have, for any $s \gg 0$, by Lemma 2.1

$$y^s x_\varphi = x_\varphi y^s + c_1^{(s)} x_{\varphi-\alpha} y^{s-1} + \ldots + c_p^{(s)} \cdot x_{\varphi - p\alpha} y^{s-p}.$$

where $c_1^{(s)}, \ldots, c_p^{(s)}$ are integers depending on the choice of the Chavalley basis and s. The integer p depends only on the 'type' of the root-system generated by φ and α. (In all cases, $p \leqq 3$.) Also, the above holds for all $s \geqq p$. We rewrite as follows:

$$\begin{aligned} y^s x_\varphi &= (x_\varphi y^p + \ldots + c_p^{(s)} x_{\varphi - p\alpha}) y^{s-p} \\ &= (y^p x_\varphi + d_1^{(s)} y^{p-1} x_{\varphi-\alpha} + \ldots + d_p^{(s)} x_{\varphi - p\alpha}) y^{s-p}. \end{aligned}$$

Thus,

$$x_\varphi y^{-n} \xi = y^{-(n+p)} (y^p \cdot x_\varphi + d_1^{(n)} y^{p-1} x_{\varphi-\alpha} + \ldots + d_p^{(n)} x_{\varphi - p\alpha}) \xi \qquad (s-p=n,\ \xi \in M).$$

So

$$\begin{aligned} x_\varphi^t y^{-n} \xi &= y^{-(n+tp)} (y^p x_\varphi + d_1^{(n+t-1)} \cdot y^{p-1} x_{\varphi-\alpha} + \ldots + d_p^{(n+t-1)} \cdot x_{\varphi - p\alpha}) \\ &\quad \ldots (y^p x_\varphi + d_1^{(n)} y^{p-1} x_{\varphi-\alpha} + \ldots + d_p^{(n)} x_{\varphi - p\alpha}) \xi \\ &= y^{-(n+tp)} \left(\sum_{i=0}^{tp} \sum_{\substack{a_j \geqq 0 \\ \Sigma a_j = t \\ \Sigma a_j \cdot j = tp - i}} b_{a_0, \ldots, a_p}^{(i)} \cdot y^i x_\varphi^{a_0} \ldots x_{\varphi - p\alpha}^{a_p} \xi \right) \end{aligned} \tag{$*$}$$

We next observe that there exists $l \geqq 1$ such that for any sequence $k_1, \ldots, k_p$ of integers $\geqq 1$, one has:

$$\begin{aligned} & x_{\varphi-(p-1)\alpha}^s \cdot x_{\varphi - p\alpha}^{k_p} = u_{p-1} x_{\varphi-(p-1)\alpha}^{s - lk_p} \qquad \text{for some } u_{p-1} \in \mathrm{U}(\mathfrak{g}),\ s \geqq lk_p \\ & x_{\varphi-(p-2)\alpha}^s \cdot x_{\varphi-(p-1)\alpha}^{k_{p-1}} \cdot x_{\varphi - p\alpha}^{k_p} = u_{p-2} x_{\varphi-(p-2)\alpha}^{s - l(k_p + k_{p-1})} \qquad \text{for } s \geqq l(k_p + k_{p-1}) \end{aligned} \tag{$**$}$$

and so on.

We have to show that $x_\varphi^r \cdot y^{-n} \xi = 0$ for $r \gg 0$. Let $q_0, \ldots, q_p$ be such that $x_{\varphi - i\alpha}^{q_i} \xi = 0\, \forall\, i\ (\xi \in M)$. Let $t \geqq q_0 + (l+1) q_1 + \ldots + (l+1)^p q_p$. We will show that $x_\varphi^t y^{-n} \xi = 0$. Intuitively this is clear from $(*)$ and $(**)$. We make it more explicit.

Take a term $y^i x_\varphi^{a_0} \ldots x_{\varphi - p\alpha}^{a_p}$ occuring in $(*)$. Now either $a_p \geqq q_p$ in which case $x_{\varphi - p\alpha}^{a_p} \xi = 0$ and this term vanishes or $a_p < q_p$. Next, either $a_{p-1} \geqq l a_p + q_{p-1}$ in which case

$$x_{\varphi-(p-1)\alpha}^{a_{p-1}} \cdot x_{\varphi - p\alpha}^{a_p} \xi = u_{p-1} \cdot x_{\varphi-(p-1)\alpha}^{a_{p-1} - l a_p} \xi = 0$$

and the term vanishes or $a_{p-1} - l a_p < q_{p-1}$ and so on.

Thus the term $y^i \cdot x_\varphi^{a_0} \ldots x_{\varphi - p\alpha}^{a_p} \xi = 0$ unless

$$a_p < q_p, \quad a_{p-1} - l a_p < q_{p-1}, \ldots, a_0 - l(a_1 + \ldots + a_p) < q_0.$$

But then

$$\begin{aligned} & a_0 < q_0 + l q_1 + l(l+1) q_2 + \ldots + l(l+1)^{p-1} q_p, \\ & a_1 < q_1 + l q_2 + \ldots + l(l+1)^{p-2} q_p \end{aligned}$$

and so on.

Thus $t=a_0+\ldots+a_p<q_0+(l+1)q_1+\ldots+(l+1)^p q_p$ which can't happen. We have thus proved that $x_\varphi^t y^{-n}\xi=0$. This completes proof of the proposition.

It is clear from the above proposition that an element $y^{-n}\xi\in D_\alpha(M)$ belongs to $C_\alpha(M)$ iff $x^t y^{-n}\xi=0$ for $t\gg 0$. In other words, the $sl_2(=\{x,y,H\})$ corresponding to α governs the difference between $D_\alpha(M)$ and $C_\alpha(M)$, a fact which is only to be expected. We now give a criterion to determine whether an element $y^{-n}m\in D_\alpha(M)$ belongs to $C_\alpha(M)$ or not. First observe that any element $\xi\neq 0\in D_\alpha(M)$ can be uniquely expressed as $y^{-n}m$, $n\geqq 0$, $m\in M$ and n is minimal with respect to this property. We call this expression as the 'minimal' expression for ξ e.g. $\xi\in M$ iff $n=0$ in the minimal expression.

Theorem 2.6. *Let $\xi\in D_\alpha(M)\setminus M$ be an element of weight $\lambda\in\mathfrak{h}^*$. Let $\xi=y^{-n}m$ be the minimal expression for ξ ($n>0$). Then $\xi\in C_\alpha(M)$ iff the following two conditions are satisfied.*

i) $n-\lambda_\alpha=j\geqq 1$ ($\lambda_\alpha=\lambda(H), j\in\mathbf{Z}$).

ii) $x^t y^{t-j}m=0$ *for some* $t\geqq j$.

Proof. We have,

$$\begin{aligned} y^{n+1}x &= x y^{n+1}-(n+1)(H+n)y^n \\ &= (xy-(n+1)(H+n))y^n \\ &= (yx-n(H+n+1))y^n. \end{aligned}$$

Hence $xy^{-n}m=y^{-(n+1)}(yx-n(H+n+1))m$. As ξ is of weight λ, m is of weight $\mu=\lambda-n\alpha$ and so $H\cdot m=\mu(H)\cdot m=(\lambda_\alpha-2n)m$. Thus $xy^{-n}m=y^{-(n+1)}(yx-n(\lambda_\alpha-n+1))m$.

Continuing in this way,

$$\begin{aligned} x^t y^{-n}m &= y^{-(n+t)}(yx-(n+t-1)(\lambda_\alpha-n+t))\cdot\ldots\cdot(yx-(n+1)(\lambda_\alpha-n+2)) \\ &\quad\cdot(yx-n(\lambda_\alpha-n+1))m \\ &= y^{-(n+1)}(ym_1+(-1)^t n(n+1) \\ &\quad\ldots(n+t-1)(\lambda_\alpha-n+1)\cdot\ldots\cdot(\lambda_\alpha-n+t)\cdot m \qquad \text{for a suitable } m_1\in M. \end{aligned}$$

Assume that $\xi\in C_\alpha(M)$ i.e. $x^t\cdot y^{-n}m=0$ for some $t\gg 0$. Thus, $ym_1+(-1)^t\cdot n\ldots(n+t-1)\cdot(\lambda_\alpha-n+1)\ldots(\lambda_\alpha-n+t)m=0$. Since $m\notin yM$ (otherwise $\xi=y^{-n}m$ would not be a minimal expression), the coefficient of m in above equation must be zero.

Thus $\lambda_\alpha=n-j$ for some $1\leqq j\leqq t$.

Also, we have,

$$\begin{aligned} x^t y^{-n}m &= y^{-(n+t)}(yx-(n+t-1)(t-j))\ldots(yx-(n+j)\cdot 1) \\ &\quad\cdot yx\cdot\ldots\cdot(yx+n(j-1))m \qquad (*) \end{aligned}$$

We remark here that the brackets in (*) commute with each other and so can be taken in any order.

Next, we have, for $s\geqq 1$,

$$\begin{aligned} y^s x^s &= y^{s-1}yx^{s-1}x = y^{s-1}(x^{s-1}y-(s-1)\cdot x^{s-2}(H+s-2))x \\ &= y^{s-1}x^{s-1}\cdot yx-(s-1)y^{s-1}x^{s-1}(H+s) \\ &= y^{s-1}x^{s-1}(yx-(H+s)). \end{aligned}$$

From this one derives immediately that for any $m'\in M$ with $H\cdot m'=am'$, one has

$$y^s x^s m' = yx(yx-(a+2)(yx-2(a+3))\cdot\ldots\cdot(yx-(s-1)(a+s))m'. \tag{**}$$

Using this in (*), we get

$$yx(yx+(n+j-2))\cdot\ldots\cdot(yx+n(j-1))m=y^j x^j m.$$

Thus

$$\begin{aligned} x^t y^{-n} m &= y^{-(n+t)}(yx-(n+t-1)(t-j))\cdot\ldots\cdot(yx-(n+j))\cdot y^j x^j m_0\\ &= y^{-(n+t)} y^j x^j (yx-(n+t-1)(t-j))\cdot\ldots\cdot(yx-(n+j))m \end{aligned}$$

(by the remark made earlier).

Use (**) again with x, y interchanged and $m'=m$, $s=t-j$ to get

$$\begin{aligned} x^{t-j} y^{t-j} m &= xy(xy-(n+j+2))\cdot\ldots\cdot(xy-(t-j-1)(n+j+t-j))m\\ &= (yx-(n+j))(yx-2(n+j+1))\cdot\ldots\cdot(yx-(t-j)(n+t-1))m. \end{aligned}$$

Thus (*) reduces to

$$x^t y^{-n} m = y^{-(n+t-j)} x^t \cdot y^{t-j} m.$$

Therefore $x^t y^{t-j} m=0$. This proves the 'necessity' part.

We observe that we have also proved the following: Under the hypothesis $n-\lambda_\alpha=j\geqq 1$, $x^t y^{-n} m = y^{-(n+t-j)}\cdot x^t\cdot y^{t-j} m$ $\forall t\geqq j$. The 'sufficiency' part is now clear.

This concludes the proof of the theorem.

Corollary 2.7. *Let $m\in M$ be a non-zero vector such that* (i) $m\notin yM$, (ii) $Hm=\mu m$ *with* $\mu\in\mathbf{Z}$, $\mu<0$ *and* (iii) $x^{-\mu}m=0$. *Then $y^{-n}m\in C_\alpha(M)$ iff $n\leqq-(\mu+k)$ where k is the least integer such that $x^k m=0$. (Note $k\leqq-\mu$.) Further, k is also the least integer such that $x^k y^{(\mu+k)} m=0$.*

In particular, if m is x-invariant (i.e. $k=1$) (along with $\mu<0$) then $y^{-n}m\in C_\alpha(M)$ iff $n\leqq-(\mu+1)$ and $y^{\mu+1}m$ is a x-invariant as well.

Proof. We prove that $y^{\mu+k-1}m\notin C_\alpha(M)$ and $y^{\mu+k}m\in C_\alpha(M)$. By the theorem above, $y^{\mu+k-1}m\in C_\alpha(M)$ iff $x^t y^{t-k+1}m=0$ for $t\gg 0$.

Since $x^k m=0$, this is equivalent to $\binom{H-k+1}{t-k+1}x^{k-1}m=0$. (We use commutation formula for $x^r\cdot y^s$ (cf. [7, § 2]).) Here,

$$\binom{H-k+1}{t-k+1}=\frac{(H-k+1)(H-k)\ldots(H-k+1-t+k-1+1)}{1\cdot 2\ldots(t-k+1)}.$$

Thus

$$\binom{H-k+1}{t-k+1}x^{k-1}m=\frac{(\mu+k-1)\ldots(\mu+k-(t-k+1))}{1\cdot 2\ldots(t-k+1)}x^{k-1}m\neq 0$$

as $x^{k-1}m\neq 0$ and $\mu+k\leqq 0$. Hence $y^{\mu+k-1}m\notin C_\alpha(M)$.

Since $x^k m=0$, it follows immediately that $x^t y^{t-k}m=0$ $\forall t\geqq k$ and so $y^{\mu+k}m\in C_\alpha(M)$.

Next, it is clear that $x^k \cdot y^{\mu+k} m = 0$ as $x^k m = 0$. We now prove that $x^{k-1} \cdot y^{\mu+k} m \neq 0$. If $\mu + k = 0$, then this is clear so assume that $\mu + k < 0$. Now we go back to various identities given in the proof of the Theorem above. We use them without mention. Thus,

$$x^{k-1} y^{\mu+k} m = y^{\mu+k-k+1}(yx + (-\mu-2)) \cdot \ldots \cdot (yx + (k-1)(-\mu-k))m.$$

Also,

$$y^s x^s m = yx(yx - (\mu+2)) \ldots (yx - (s-1)(\mu+s))m \qquad \text{for } s \geqq 0.$$

So

$$x^{k-1} y^{\mu+k} m = y^{\mu+1}(y^{k-1} x^{k-1} m - (k-1)(\mu+k) y^{k-2} x^{k-2} m + \ldots \\ + (-1)^{k-1}(k-1) \ldots 2 \cdot 1 \cdot (\mu+k) \ldots (\mu+2) \cdot m).$$

Note that the coefficient of $m = (-1)^{k-1} \cdot k-1!\,(\mu+k) \ldots (\mu+2) \neq 0$ as $\mu + k < 0$. So $x^{k-1} y^{\mu+k} m = 0 \Rightarrow m \in yM$ which is not true. Hence $x^{k-1} y^{\mu+k} m \neq 0$.

The particular case of $k = 1$ is quite clear.

Remark. If one drops the condition $\mu + k \leqq 0$ (i.e. $x^{-\mu} m = 0$) then in general one can't say anything about the biggest n such that $y^{-n} m \in C_\alpha(M)$.

§ 3. Tensoring with Finite Dimensional $\mathfrak{g}$-modules

Although finite dimensional $\mathfrak{g}$-modules F do not belong to $\mathscr{A}$, the tensor product $F \otimes A$ with $A \in \mathscr{A}$ belongs to $\mathscr{A}$ as can be easily verified. This tensoring with finite dimensional modules has turned out to be a very useful technique in representation theory (e.g. [9]). We are interested in finding out how our α-localization functor behaves with respect to this tensoring. We have

Theorem 3.1. *If F is a finite dimensional $\mathfrak{g}$-module and $A \in \mathscr{A}$, then $\exists$ a canonical $\mathfrak{g}$-module isomorphism g: $D_\alpha(F \otimes A) \simeq F \otimes D_\alpha(A)$.*

Proof. Any element ξ of $D_\alpha(F \otimes A)$ is of the form

$$\xi = y^{-n}\Big(\sum_i e_i \otimes a_i\Big), \qquad n \geqq 0,\ e_i \in F,\ a_i \in A.$$

Define

$$g(\xi) = \sum_i \sum_{r=0}^{\infty} (-1)^r \binom{r+n-1}{r} (y^r e_i \otimes y^{-n-r} a_i).$$

Note. (i) The right hand side is a finite sum as $y^r e_i = 0$ for $r \gg 0$.

(ii) The bionomial coefficient $\binom{m}{s}$ for $m, s \in \mathbf{Z}$ is defined by

$$(1+x)^m = \sum_{s=0}^{\infty} \binom{m}{s} x^s.$$

We have to show that g is well-defined.

So let

$$\xi = y^{-n}(\sum_i e_i \otimes a_i) = y^{-k}(\sum_j f_j \otimes b_j) \qquad f_j \in F,\ b_j \in A$$

i.e. $y^k(\sum_i e_i \otimes a_i) = y^n(\sum_j f_j \otimes b_j)$. Consider

$$y^{n+k}\left(\sum_i \sum_{r=0}^{\infty} (-1)^r \cdot \binom{r+n-1}{r} (y^r e_i \otimes y^{-n-r} a_i)\right.$$

$$= \sum_i \sum_{r=0}^{\infty} (-1)^r \binom{r+n-1}{r} \left(\sum_{t=0}^{\infty} \binom{n+k}{t} (y^{t+r} e_i \otimes y^{n+k-t-n-r} a_i)\right.$$

$$\left(\text{Note: } \binom{n+k}{t} = 0 \text{ for } t > n+k\right)$$

$$= \sum_i \sum_{l=0}^{\infty} \left(\sum_{r=0}^{l} (-1)^r \binom{r+n-1}{r} \binom{n+k}{l-r}\right) (y^l e_i \otimes y^{k-l} a_i)\Big)$$

putting $t + r = l$.

Now,

$$(-1)^r \binom{r+n-1}{r} = \binom{-n}{r}$$

so

$$\sum_{r=0}^{l} (-1)^r \binom{r+n-1}{r} \binom{n+k}{l-r} = \sum_{r=0}^{l} \binom{-n}{r} \binom{n+k}{l-r} = \binom{k}{l}.$$

(The last equality follows from $(1+x)^{n+k} \cdot (1+x)^{-n} = (1+x)^k$ by comparing the coefficients of x on both the sides.)

Thus,

$$y^{n+k} \sum_i \sum_{r=0}^{\infty} (-1)^r \binom{r+n-1}{r} (y^r e_i \otimes y^{-n-r} a_i)$$

$$= \sum_i \sum_{l=0}^{\infty} \binom{k}{l} (y^l e_i \otimes y^{k-l} a_i)$$

$$= y^k (\sum_i e_i \otimes a_i)$$

$$= y^n (\sum_j f_j \otimes b_j)$$

$$= y^{n+k} \sum_j \sum_{t=0}^{\infty} (-1)^t \binom{t+k-1}{t} (y^t f_j \otimes y^{-k-t} b_j)$$

(by an argument similar to one above).

Since $F \otimes D_\alpha(A)$ is $U(\mathfrak{n}^-)$-torsion free, we have prored that g is well-defined.

We next prove that g is a $\mathfrak{g}$-module homomorphism. For this observe the following:

(i) $$g(\sum_i e_i \otimes a_i) = \sum_i \sum_{r=0}^{\infty} (-1)^r \binom{r-1}{r} (y^r e_i \otimes y^{-r} a_i)$$

$$= \sum_i e_i \otimes a_i \quad \text{as} \quad \binom{r-1}{r} = \delta_{r,0}.$$

(ii) For any $\xi = y^{-n}(\sum_i e_i \otimes a_i) \in D_\alpha(F \otimes A)$,

$$\begin{aligned} g(y \cdot \xi) &= g(y^{-n}(\sum_i (y e_i \otimes a_i + e_i \otimes y a_i))) \\ &= \sum_i \sum_{r=0}^{\infty} (-1)^r \binom{r+n-1}{r} (y^{r+1} e_i \otimes y^{-n-r} a_i + y^r e_i \otimes y^{-n+1-r} a_i) \\ &= y\left(\sum_i \sum_{r=0}^{\infty} (-1)^r \binom{r+n-1}{r} (y^r e_i \otimes y^{-n-r} a_i)\right) \\ &= y g(\xi). \end{aligned}$$

Now let $z \in \mathfrak{g}$, $\xi = y^{-n}(\sum_i e_i \otimes a_i) \in D_\alpha(F \otimes A)$ be arbitrary. Choose $s \geqq 1$ such that $y^s z = u \cdot y^n$ for some $u \in U(\mathfrak{g})$. (Such s exists by Lemma 2.1.)

We have,

$$\begin{aligned} y^s g(z \cdot \xi) &= g(y^s z \xi) \quad \text{by (ii) above} \\ &= g(u \cdot y^n \cdot y^{-n}(\sum_i e_i \otimes a_i)) \\ &= g(u(\sum_i e_i \otimes a_i)) \\ &= u(\sum_i e_i \otimes a_i) \quad \text{by (i) above.} \end{aligned}$$

Also,

$$\begin{aligned} y^s z \cdot g(\xi) &= u y^n \cdot g(\xi) \\ &= u \cdot g(y^n \xi) = u \cdot g(\sum_i e_i \otimes a_i) \\ &= u(\sum_i e_i \otimes a_i). \end{aligned}$$

Since $F \otimes D_\alpha(A)$ is $U(\mathfrak{n}^-)$-torsion-free, we have proved that $g(z \cdot \xi) = z \cdot g(\xi)$ i.e. g is a $\mathfrak{g}$-module homomorphism.

Next, define f: $F \otimes D_\alpha(A) \to D_\alpha(F \otimes A)$ as follows:

$$f(\sum_i e_i \otimes y^{-n_i} a_i) = \sum_i y^{-k_i t} \left(\sum_{r=0}^{\infty} \binom{k_i t}{r} (y^r e_i \otimes y^{k_i t - n_i} a_i)\right)$$

where t is such that $y^t(F) = 0$ and k_i's are such that $(k_i - 1)t \geqq n_i - 1$. As $y^t(F) = 0$, the summation on r on the right hand side is upto $t-1$ only and in that case $k_i t - n_i - r \geqq k_i t - n_i - t + 1 = (k_i - 1)t - n_i + 1 \geqq 0$ and so

$$\sum_{r=0}^{\infty} \binom{k_i t}{r} (y^r e_i \otimes y^{k_i t - n_i - r} a_i) \in F \otimes A.$$

It is quite straight forward to show (using computations similar to those given above) that f is well-defined and that f and g are inverses of each other. We omit the details. Since g is a $\mathfrak{g}$-module homomorphism, so is f. This proves the theorem.

Remark. For the sake of completeness, we give the correct definition of the bionomial coefficients. For $m, r \in \mathbf{Z}$, define $\binom{m}{r}$ by

$$(1+x)^m = \sum_{s \in \mathbf{Z}} \binom{m}{s} x^s.$$

Then

(i) $\binom{0}{r} = \delta_{0,r} \quad \forall\, r,$

(ii) $\binom{m}{r} = 0 \quad \text{for } r \leqq -1,$

(iii) $\binom{m}{0} = 1 \; \forall\, m,$

(iv) $\binom{m}{r} = \frac{m(m-1)\dots(m-r+1)}{1\dots r} \quad \forall\, r \geqq 1, \; m \in \mathbf{Z}.$

Using the identity $(1+x)^m(1+x)^n = (1+x)^{m+n}$ $\forall\, m, n \in \mathbf{Z}$, we get various identities between the bionomial coefficients e.g. $\sum_{r=0}^{l} (-1)^r \cdot \binom{r+n-1}{r} \binom{n+k}{-r} = \binom{k}{l}$ as used above.

Corollary 3.2. *If $M \in \mathscr{I}$ and F is a finite dimensional $\mathfrak{g}$-module then $F \otimes M \in \mathscr{I}$ again and $\exists$ a canonical $\mathfrak{g}$-module isomorphism g:*

$$C_\alpha(F \otimes M) \simeq F \otimes C_\alpha(M).$$

Proof. Consider the $\mathfrak{g}$-module isomorphism $g: D_\alpha(F \otimes M) \simeq F \otimes D_\alpha(M)$. It is enough to show that

$$F \otimes C_\alpha(M) = K,$$

where K is the submodule

$$\{\xi \in F \otimes D_\alpha(M) \mid x_\varphi \text{ acts locally nilpotently on } \xi \; \forall\, \varphi \in \Phi^+\}.$$

Let $\{e_1 \dots e_n\}$ be a basis of F such that e_i is of weight $\lambda_i \in \mathfrak{h}^*$, the labelling being so arranged that $\lambda_i - \lambda_j = \sum_{\beta \in \Delta} a_\beta \cdot \beta$ with $a_\beta \geqq 0 \Rightarrow j < i$.

Let $\xi = \sum_i e_i \otimes \xi_i \in K$ and $\varphi \in \Phi^+$. Then $x_\varphi^t \xi = 0$ for $t \gg 0$. So

$$0 = \sum_i \sum_{j=0}^{t} \binom{t}{j} (x_\varphi^j e_i \otimes x_\varphi^{t-j} \xi_i)$$

$$= e_1 \otimes x_\varphi^t \xi_1 + \sum_{i \geqq 2} e_i \otimes \xi_i' \qquad \text{for suitable } \xi_i'.$$

Thus $x_\varphi^t \xi_1 = 0$. One can now proceed by induction on i and show that x_φ acts locally nilpotently on $\xi_i \forall_i$. Thus $\xi_i \in C_\alpha(M)$ and $\xi \in F \otimes C_\alpha(M)$. The implication $\xi \in F \otimes C_\alpha(M) \Rightarrow \xi \in K$ is even easier. This completes the proof of the corollary.

We are now in a position to extend the notion of completions in the following set up (cf. [3, § 3]). Let $\mathfrak{g}, \mathscr{A}, \mathscr{I}, \alpha$ be as before. Let $\mathfrak{m} \supseteq \mathfrak{g}$ be a finite dimensional ambient Lie algebra. Let $\mathscr{A}_\mathfrak{g}^\mathfrak{m}$ (respectively $\mathscr{I}_\mathfrak{g}^\mathfrak{m}$) be the category of $\mathfrak{m}$-modules which when considered as $\mathfrak{g}$-modules belong to $\mathscr{A}$ (respectively $\mathscr{I}$). We now have

Proposition 3.3. *Let $A \in \mathscr{A}_\mathfrak{g}^\mathfrak{m}$ (respectively $M \in \mathscr{I}_\mathfrak{g}^\mathfrak{m}$), then $D_\alpha(A) \in \mathscr{A}_\mathfrak{g}^\mathfrak{m}$ (respectively $C_\alpha(M) \in \mathscr{I}_\mathfrak{g}^\mathfrak{m}$).*

Proof. $\mathfrak{m}$ is a finite dimensional $\mathfrak{g}$-module under the adjoint action so by Theorem 3.1, we have $\mathfrak{m} \otimes D_\alpha(A) \simeq D_\alpha(\mathfrak{m} \otimes A)$.

Consider the map $\mathfrak{m} \otimes A \to A$ given by the $\mathfrak{m}$-action. This is a $\mathfrak{g}$-module map and so gives rise to a map $D_\alpha(\mathfrak{m} \otimes A) \to D_\alpha(A)$ by functoriallity of D_α. Thus we have a map $\mathfrak{m} \otimes D_\alpha(A) \to D_\alpha(A)$. We take this to be the $\mathfrak{m}$-action on $D_\alpha(A)$. It is easy to verify that this is indeed a Lie algebra action. The statement about $\mathscr{I}_\mathfrak{g}^\mathfrak{m}$ and the functor C_α follows similarly using Corollary 3.2.

Remark 1. The idea of the proof of this proposition is essentially due to M.S. Raghunathan.

Remark 2. The principal application of this 'ambient' setup is to the case when $\mathfrak{m}$ and $\mathfrak{g}$ are complexifications of $\mathfrak{m}_0$ and $\mathfrak{g}_0$ respectively where $\mathfrak{m}_0$ is a real semisimple Lie algebra and $\mathfrak{g}_0$ is its maximal compact subalgebra. Successive completions of $\mathfrak{m}$-modules in $\mathscr{I}_\mathfrak{g}^\mathfrak{m}$ with respect to *simple compact* roots give rise to Harish Chandra $\mathfrak{m}$-modules and it is the central theme of Enright's work ([3, 4]).

We come to another application of Theorem 3.1 which brings out an interesting (and somewhat mysterious) aspect of the structure of D_α's. First note that the Verma module V_{λ,Φ^+} $(\lambda \in \mathfrak{h}^*)$ (see [2] for an exposition) is in the category $\mathscr{I}$.

Proposition 3.4. *Let V_{λ,Φ^+} be the Verma module of highest weight $\lambda \in \mathfrak{h}^*$ such that $\lambda(H_\alpha) = \lambda_\alpha \notin \{-2, -3, \ldots\}$. Then one has a non-split exact sequence*

$$O \to V_{\lambda,\Phi^+} \to D_\alpha(V_{\lambda,\Phi^+}) \to V_{\lambda+\alpha, s_\alpha(\Phi^+)} \to 0$$

where s_α is the reflection with respect to α and $s_\alpha(\Phi^+)$ is the corresponding positive root system.

Proof. We first note that the condition $\lambda(H_\alpha) \notin \{-2, -3, \ldots\}$ precisely means that $C_\alpha(V_{\lambda,\Phi^+}) = V_{\lambda,\Phi^+}$. (We give a proof of this fact later on.) Therefore for $\xi = y^{-n} m \in D_\alpha(V_{\lambda,\Phi^+}) \setminus V_{\lambda,\Phi^+}$, $x^r \cdot \xi \neq 0 \;\forall r \geqq 0$.

Consider the $\mathfrak{g}$-module $D_\alpha(V_{\lambda,\Phi^+})/V_{\lambda,\Phi^+}$; let π be the natural projection from $D_\alpha(V_{\lambda,\Phi^+})$ onto it. Let v_λ be a generator of V_{λ,Φ^+}. We claim that $\pi(y^{-1} v_\lambda) \neq 0$ and it is a highest weight vector with respect to $s_\alpha(\Phi^+)$. Clearly $y^{-1} v_\lambda \notin V_{\lambda,\Phi^+}$ for weight-considerations and so $\pi(y^{-1} \cdot v_\lambda) \neq 0$.

Let $\varphi \in \Phi^+$, $\varphi \neq \alpha$. We have (see Proposition 2.5),

$$x_\varphi y^{-1} v_\lambda = y^{-(1+p)} \cdot (y^p x_\varphi + d_1^{(1)} y^{p-1} x_{\varphi-\alpha} + \ldots + d_p^{(1)} x_{\varphi - p\alpha}) v_\lambda = 0.$$

Also, $y \cdot \pi(y^{-1} v_\lambda) = \pi(v_\lambda) = 0$. This proves the above claim.

We next prove that $y^{-1}v_\lambda$ generates $D_\alpha(V_{\lambda,\Phi^+})$. So let K be the $\mathfrak{g}$-module generated by $y^{-1}v_\lambda$. Clearly $K\supset V_{\lambda,\Phi^+}$. Next we prove by induction on r that $y^{-r}v_\lambda\in K$. For $r=1$, it is clear. Let $r\geqq 2$. We have,

$$xy^{-(r-1)}v_\lambda=y^{-r}(yx-(r-1)(H+r))v_\lambda=-(r-1)(\lambda_\alpha+r)\cdot y^{-r}v_\lambda.$$

As $\lambda_\alpha\notin\{-2,-3,\ldots\}$, $\lambda_\alpha+r\neq 0\ \forall\, r\geqq 2$. Thus $y^{-r}v_\lambda\in K$ as $y^{-(r-1)}v_\lambda\in K$ by induction hypothesis.

For a root $\varphi\in\Phi^+$, define $l(\varphi)=\max\limits_t\{\varphi+t\alpha\in\Phi\}$. For a monomial $u_0=y_{\varphi_1}\ldots y_{\varphi_k}$, define $l(u_0)=\sum\limits_{i=1}^{k} l(\varphi_i)$. For an arbitrary $u\in\mathrm{U}(\mathfrak{a})$, define $l(u)=\max l(u_0)$, u_0 is a monomial occuring u.

By, convention, $l(0)=0$.

We prove by induction on $l(u)$ that $y^{-n}uv_\lambda\in K\ \forall\, n\geqq 0$. If $l(u)=0$, then $yu=uy$ and $y^{-n}uv_\lambda=u\cdot y^{-n}v_\lambda\in K$. Let $l(u)\geqq 1$. Given $n\geqq 1$, $\exists\, s\geqq 1$ such that $y^s u=uy^s+u_1y^n$ with $l(u_1)<l(u)$. (This is really Lemma 2.1; we keep track of the various terms occuring the expression on the right hand side of it.) Further $y^s u=uy^s+u_1y^n=(y^{s-n}u+u_2+u_1)y^n$ with $l(u_2)<l(u)$. So $uy^{-n}v_\lambda=y^{-s}(y^{s-n}u+u_2+u_1)v_\lambda=y^{-n}uv_\lambda+y^{-s}(u_2+u_1)v_\lambda$. By induction, $y^{-s}(u_2+u_1)v_\lambda\in K$. Also $u\cdot y^{-n}v_\lambda\in K$ and so $y^{-n}u\cdot v_\lambda\in K$. This proves that $K=D_\alpha(V_{\lambda,\Phi^+})$. Also, as the weight space corresponding to weight $\lambda+\alpha$ is one-dimensional and contains generator of $D_\alpha(V_{\lambda,\Phi^+})$, it is clear that $D_\alpha(V_{\lambda,\Phi^+})$ is indecomposable.

Now $D_\alpha(V_{\lambda,\Phi^+})/V_{\lambda,\Phi^+}$ is a highest weight module with respect to $s_\alpha(\Phi^+)$ and is generated by $\pi(y^{-1}v_\lambda)$. We now claim that it is actually isomorphic to $V_{\lambda+\alpha,s_\alpha(\Phi^+)}$. This is achieved by computing the firmal character of $D_\alpha(V_{\lambda,\Phi^+})/V_{\lambda,\Phi^+}$ and showing it to be equal to that of $V_{\lambda+\alpha,s_\alpha(\Phi^+)}$. Consider $\mathfrak{r}_\alpha=\mathfrak{h}\oplus\mathfrak{g}^\alpha\oplus\mathfrak{g}^{-\alpha}$; $\mathfrak{r}_\alpha$ is reductive of semisimple rank 1.

If $A_{\lambda_\alpha}=\{v_\lambda, yv_\lambda,\ldots\}$, then A_{λ_α} is a $\mathfrak{r}_\alpha$-highest weight module of highest weight λ_α.

Also, $V_{\lambda,\Phi^+}\simeq\mathrm{U}(\mathfrak{n}^--\{\mathfrak{g}^{-\alpha}\})\otimes A_{\lambda_\alpha}$ as $\mathfrak{r}_\alpha$-modules; $\mathrm{U}(\mathfrak{n}^--\{\mathfrak{g}^{-\alpha}\})$ is a $\mathfrak{r}_\alpha$-module under adjoint action and one knows that $\mathrm{U}(\mathfrak{n}^--\{\mathfrak{g}^{-\alpha}\})\simeq\bigoplus\limits_i F_i$, F_i's are finite dimensional $\mathfrak{r}_\alpha$-modules.

Thus, $V_{\lambda,\Phi^+}\underset{\eta}{\simeq}\bigoplus\limits_i F_i\otimes A_{\lambda_\alpha}$ as $\mathfrak{r}_\alpha$-modules. (Note that $\lambda_\alpha\notin\{-2,-3,\ldots\}$ immediately implies that A_{λ_α} is complete with respect to α and as C_α behaves well with finite dimensional tensoring (Corollary 3.2) and direct sums, V_{λ,Φ^+} is complete as well.)

The functor D_α commutes finite dimensional tensoring (Theorem 3.1) and so

$$D_\alpha(V_{\lambda,\Phi^+})\underset{D_\alpha(\eta)}{\simeq}\bigoplus_i D_\alpha(F_i\otimes A_{\lambda_\alpha})\simeq\bigoplus_i F_i\otimes D_\alpha(A_{\lambda_\alpha}).$$

Also, $V_{\lambda,\Phi^+}\simeq\bigoplus\limits_i F_i\otimes A_{\lambda_\alpha}$.

Since all the maps are canonical,

$$D_\alpha(V_{\lambda,\Phi^+})/V_{\lambda,\Phi^+}\simeq\bigoplus_i F_i\otimes D_\alpha(A_{\lambda_\alpha})/A_{\lambda_\alpha}\quad\text{as } \mathfrak{r}_\alpha\text{-modules}$$

$$\simeq\mathrm{U}(\mathfrak{n}^--\{\mathfrak{g}^{-\alpha}\})\otimes(D_\alpha(A_{\lambda_\alpha})/A_{\lambda_\alpha}).$$

Consider the formal characters.

$$\operatorname{ch}(D_\alpha(V_{\lambda,\Phi^+})/V_{\lambda,\Phi^+}) = \operatorname{ch}(U(\mathfrak{n}^- - \{\mathfrak{g}^{-\alpha}\}))\cdot \operatorname{ch}(D_\alpha(A_{\lambda_\alpha})/A_{\lambda_\alpha})$$
$$= \frac{1}{\prod_{\substack{\varphi\in\Phi^+\\ \varphi\neq\alpha}}(1-e^{-\varphi})}\cdot \operatorname{ch}(D_\alpha(A_{\lambda_\alpha})/A_{\lambda_\alpha}).$$

Now $D_\alpha(A_{\lambda_\alpha})/A_{\lambda_\alpha}$ is a $\mathfrak{r}_\alpha$-highest weight modules generated by $\pi(y^{-1}v_\lambda)$ (as seen before). Also, $\pi(xy^{-r}v_\lambda)=0 \Rightarrow \pi(-r(r+1+\lambda_\alpha)\cdot y^{-(r+1)}v_\lambda)=0 \Rightarrow r=0$ as $y^{-(r+1)}v_\lambda \notin A_{\lambda_\alpha}$ for $r\geq 1$. Thus $D_\alpha(A_{\lambda_\alpha})/A_{\lambda_\alpha} \simeq$ Verma module for $\mathfrak{r}_\alpha$ with respect to the positive root system $\{-\alpha\}$.

Hence

$$\operatorname{ch} D_\alpha(A_{\lambda_\alpha})/A_{\lambda_\alpha} = e^{\lambda+\alpha}+e^{\lambda+2\alpha}+\dots$$
$$=\frac{e^{\lambda+\alpha}}{(1-e^\alpha)}.$$

Thus

$$\operatorname{ch}(D_\alpha(V_{\lambda,\Phi^+})/V_{\lambda,\Phi^+}) = \frac{1}{\prod_{\substack{\varphi\in\Phi^+\\ \varphi\neq\alpha}}(1-e^{-\varphi})}\cdot\frac{e^{\lambda+\alpha}}{(1-e^\alpha)}$$
$$=\frac{e^{\lambda+\alpha}}{\prod_{\psi\in s_\alpha(\Phi^+)}(1-e^{-\psi})}$$
$$=\operatorname{ch}(V_{\lambda+\alpha,s_\alpha(\Phi^+)}).$$

This shows that $D_\alpha(V_{\lambda,\Phi^+})/V_{\lambda,\Phi^+} \simeq V_{\lambda+\alpha,s_\alpha(\Phi^+)}$. Thus one has an exact sequence

$$0\to V_{\lambda,\Phi^+}\to D_\alpha(V_{\lambda,\Phi^+}) \xrightarrow{\ \pi\ } V_{\lambda+\alpha,s_\alpha(\Phi^+)}\to 0.$$

This is non-split as $D_\alpha(V_{\lambda,\Phi^+})$ is indecomposable. This concludes the proof of the proposition.

General Remarks on §2, 3

Remark 1. The explicit nature of $D_\alpha(M)$'s and $C_\alpha(M)$'s enables as to simplify the proofs of many propositions in [3].

Remark 2. The construction of D_α's can be carried out in the setup of the infinite-dimensional or G.C.M. or Kăc-Moody Lie algebras (cf. [6] for an exposition of these Lie algebras.) An analogue or Theorem 3.1 with the so-called 'standard'-modules in place of finite dimensional modules is true. (Note that all we need is the fact that y acts locally nilpotently.) We plan to take up this case in a forthcoming paper.

§4. On a Problem of Enright

We have so far considered the completions with respect to a fixed simple root α. One now considers successive completions with respect to several simple roots.

Let $w \in W$, the Weyl group of $\mathfrak{g}$. Take a reduced expression $w = s_{\alpha_1} \ldots s_{\alpha_k}$ and $M \in \mathscr{I}$. One can then consider $C_{\alpha_1}(C_{\alpha_2} \ldots (C_{\alpha_k}(M)) \ldots))$. The problem posed by Enright is whether this module ($\in \mathscr{I}$) depends only on w. i.e. Is the following true:

$$\text{If } w = s_{\alpha_1} \ldots s_{\alpha_k} = s_{\beta_1} \ldots s_{\beta_k} \qquad (*)$$

are two reduced expressions then $\exists$ a $\mathfrak{g}$-module isomorphism f: $C_{\alpha_1}(\ldots(C_{\alpha_k}(M)\ldots) \simeq C_{\beta_1}(C_{\beta_2} \ldots (C_{\beta_k}(M)\ldots))$ such that f/M is identity. (It is not difficult to prove (e.g. [3, §4]) that Verma modules and their tensor products with finite dimensional modules satisfy $(*)$.)

Using the explicit construction of completions as given in §2, we prove that for any $M \in \mathscr{I}$ $(*)$ is true. We first have the following reduction:

Proposition 4.1. *If* $(*)$ *holds for all* $M \in \mathscr{I}$ *and all* $w \in$rank-2 *subgroups of* W *then* $(*)$ *holds for all* M *and all* $w \in W$.

Proof. We proceed by induction of $l(w)$. If $l(w)=1$, there is nothing to prove.

Let $w = s_{\alpha_1} \ldots s_{\alpha_k} = s_{\beta_1} \ldots s_{\beta_k}$ be two reduced expressions ($k \geqq 2$). Let $J = \{\alpha_1, \beta_1\}$. (We may assume that $\alpha_1 \neq \beta_1$). Consider the decomposition $W = W_J \cdot W^J$ (cf. [1, §3]) where W_J is the rank-2 subgroup generated by J and W^J is the set of minimal (right) coset representatives of W. Let $w = w_J \cdot \sigma$ be the corresponding decomposition. Also, one has $s_{\alpha_1} w = w'_J \cdot \sigma$; $s_{\beta_1} w = w''_J \cdot \sigma$ for some w'_J, $w''_J \in W_J$ (Note the same $\sigma \in W^J$ occurs for w, $s_{\alpha_1} w$ and $s_{\beta_1} w$.) Let $w'_J = s_{r_1} \ldots s_{r_p}$, $w''_J = s_{\delta_1} \ldots s_{\delta_p}$ and $\sigma = s_{\theta_1} \ldots s_{\theta_t}$ be reduced expressions ($r_i, \delta_j \in J$). Then we have,

$$s_{\alpha_1} w = s_{\alpha_2} \ldots s_{\alpha_k} = s_{r_1} \ldots s_{r_p} \cdot s_{\theta_1} \ldots s_{\theta_t} \quad \text{and} \quad l(s_{\alpha_1} w) < l(w).$$

Hence be induction hypothesis,

$$C_{\alpha_2}(C_{\alpha_3} \ldots (C_{\alpha_k}(M) \ldots)) \simeq C_{r_1}(C_{r_2}(\ldots(C_{r_p}(C_{\theta_1} \ldots (C_{\theta_t}(M)) \ldots))$$

so that

$$C_{\alpha_1}(C_{\alpha_2} \ldots (C_{\alpha_k}(M)) \ldots)) \simeq C_{\alpha_1}(C_{r_1}(\ldots\, C_{r_p}(C_{\theta_1} \ldots (\ldots\, C_{\theta_t}(M) \ldots)).$$

Working with $s_{\beta_1} w$, we get similarly,

$$C_{\beta_1}(C_{\beta_2} \ldots\, C_{\beta_k}(M) \ldots)) \simeq C_{\beta_1}(C_{\delta_1} \ldots\, C_{\delta_p}(C_{\theta_1} \ldots\, C_{\theta_t}(M) \ldots)).$$

Let $N = C_{\theta_1}(\ldots(C_{\theta_t}(M) \ldots)) \in \mathscr{I}$. Now $w_J \in$rank-2 subgroup W_J and has reduced expressions:

$$w_J = s_{\alpha_1} s_{\gamma_1} \ldots s_{r_p} = s_{\beta_1} s_{\delta_1} \ldots s_{\delta_p}$$

and so by hypothesis

$$C_{\alpha_1}(C_{r_1} \ldots (C_{r_p}(N)) \ldots)) \simeq C_{\beta_1}(C_{\delta_1} \ldots (C_{\delta_p}(N)) \ldots)).$$

This completes the induction hypothesis and hence the proposition is proved.

We now have:

Theorem 4.2. *For any $w \in W$, $M \in \mathscr{J}$ and two reduced expressions*

$$w = s_{\alpha_1} \dots s_{\alpha_k} = s_{\beta_1} \dots s_{\beta_k}, \qquad C_{\alpha_1}(C_{\alpha_2} \dots (C_{\alpha_k}(M)) \dots)) \simeq C_{\beta_1}(\dots (C_{\beta_k}(M)) \dots))$$

in such a way that it induces identify on M.

Proof. By Proposition 4.1, one can assume that $w \in$rank-2 subgroup of W; say W_J where $J = \{\alpha, \beta\} \subseteq \Delta$.

We have to show that

$$\underbrace{C_\alpha(C_\beta \dots}_{p\text{times}}(M)) \simeq \underbrace{C_\beta(C_\alpha \dots}_{p\text{times}}(M))$$

where $p \in \{2, 3, 4, 6\}$.

We will show that $K = \underbrace{C_\alpha(C_\beta \dots}_{p\text{times}}(M))$ is complete with respect to β.

Assuming this, it is easy to complete the proof of the theorem as follows. As

$$\underbrace{C_\alpha(C_\beta \dots}_{p-1\text{times}}(M)) \hookrightarrow \underbrace{C_\alpha(C_\beta \dots}_{p\text{times}}(M)) = K$$

we get a map $\underbrace{C_\beta(C_\alpha \dots}_{p\text{times}}(M)) \hookrightarrow C_\beta(K) = K$. Similarly we have a map in the other direction and this clearly proves the isomorphism required.

To show that K is complete with respect to β, we proceed by 'first principles'. Let $\xi \in K[-s-2]x_\beta$, $s \geqq 0$. Then $\xi = y_\alpha^{-a_1} y_\beta^{-a_2} \xi_1$ where $\xi_1 \in \underbrace{C_\alpha(C_\beta \dots}_{p-2\text{times}}(M))$ and the above is a minimal expression. We may assume that ξ_1 is a weight vector (of weight $\lambda_1 \in \mathfrak{h}^*$) and that $a_1 a_2 \neq 0$ (otherwise there is nothing to prove.) We have, $-s-2 = \lambda_1(H_\beta) + 2a_2 + a_1 \cdot \alpha(H_\beta)$. Since ξ is an x_β-invariant and $[x_\beta, y_\alpha] = 0$, $y_\beta^{-a_2} \xi_1$ is also an x_β-invariant. Hence by sl_2-theory and minimality of a_2, we have:

$$x_\beta \cdot \xi_1 = 0 \quad \text{and} \quad a_2 - 1 = \lambda_1(H_\beta) + 2a_2.$$

Thus $-s-2 = a_2 - 1 + a_1 \cdot \alpha(H_\beta)$ or $-(s+1) = a_1 \cdot \alpha(H_\beta) + a_2$. We have therefore to show that $\hat{\xi} = y_\beta^{a_1 \cdot \alpha(H_\beta) + a_2} \cdot \xi$ which is an element of $D_\beta(K)$ to start with is actually in K. (This will show that K is complete with respect to β.) We now claim

$$\hat{\xi} = y_\beta^{a_1 \cdot \alpha(H_\beta) + a_2} \cdot \xi \in \underbrace{D_\alpha(D_\beta \dots}_{p\text{times}}(M)). \tag{$**$}$$

If we grant this claim then it is an easy matter to show that $\hat{\xi}$ is actually in $\underbrace{C_\alpha(C_\beta \dots}_{p\text{times}}(M)) = K$. For this, we proceed as follows: Let $\hat{\xi} = y_\alpha^{-b_1} y_\beta^{-b_2} \dots y_r^{-b_p} m$, a minimal expression with $m \in M$. (Note that this is given by ($**$).) Here, $r = \beta$ if p even and $r = \alpha$ if p odd. Let p be even i.e. $r = \beta$. Since $\hat{\xi}$ is an x_β-invariant and the above is a minimal expression, we get that $y_\beta^{-b_p} m$ is an x_β-invariant and so is in $C_\beta(M)$. Next, x_α acts nilpotently on ξ and so on $\hat{\xi} \cdot ([x_\alpha, y_\beta] = 0)$. Hence x_α acts nilpotently on $y_\beta^{b_{p-2}} \dots y_\alpha^{b_1} \hat{\xi} = y_\alpha^{-b_{p-1}} y_\beta^{-b_p} m$. Thus $y_\alpha^{-b_{p-1}} y_\beta^{-b_p} m \in C_\alpha(C_\beta(M))$.

(Note that by Proposition 2.5, one need consider the action of x_α only.) Proceeding in this way, we prove that $\hat{\xi}\in\underbrace{C_\alpha(C_\beta\dots}_{p\text{time}}(M))=K$.

The proof for p odd is similar.

We are therefore reduced to proving (**). Unfortunately we are now forced to proceed case by case. If $(\alpha,\beta)=0$ (i.e. J is of type $A_1\times A_1$) (**) is obvious. The proofs in the remaining cases are similar to each other. We give here the proof of (**), only in the most complicated case namely when J is of type G_2.

Let α be the shorter root i.e. $\alpha(H_\beta)=-1$, $\beta(H_\alpha)=-3$. Let $\xi=y_\alpha^{-a_1}y_\beta^{-a_2}y_\alpha^{-a_3}y_\beta^{-a_4}y_\alpha^{-a_5}\hat{m}$, a minimal expression with $\hat{m}\in C_\beta(M)$ and $a_i>0\,\forall i$ (otherwise there is nothing to prove). Then

$$\begin{aligned}\hat{\xi}&=y_\beta^{a_1\alpha(H_\beta)+a_2}\xi\\&=y_\beta^{-a_1+a_2}y_\alpha^{-a_1}y_\beta^{-a_2}\xi_1=y_\beta^{-a_1+a_2}y_\alpha^{-a_1}y_\beta^{-a_2}y_\alpha^{-a_3}y_\beta^{-a_4}y_\alpha^{-a_5}\hat{m}.\end{aligned}$$

The idea is to use the following identity in $U(\mathfrak{n}^-)$: For $p,q\in\mathbf{N}\cup\{0\}$,

$$y_\beta^p y_\alpha^{3p+q}y_\beta^{2p+q}y_\alpha^{3p+2q}y_\beta^{p+q}y_\alpha^q=y_\alpha^q y_\beta^{p+q}y_\alpha^{3p+2q}y_\beta^{2p+q}y_\alpha^{3p+q}y_\beta^p.$$

(Identities of this type were first observed by D.N. Verma [8]; They can be proved easily by using inclusions in Verma modules and the Poincaré-Birkoff-Witt theorem.)

However in our case, the a_i's may not be in the form required in the identity and so we change them suitably so that the new ones are in that form. The remainder of the proof shows how this can be done systematically step by step.

Let $t>0$ be such that $x_\alpha^t\xi=0$ (such t exists as $\xi\in\underbrace{C_\alpha(C_\beta\dots}_{6\text{times}}(M))$).

By Theorem 2.6, we have

$$t\geqq a_1-(\lambda_1(H_\alpha)-3a_2+2a_1)=j_1\geqq 1 \qquad \dots\text{(i)}$$

and

$$x_\alpha^t y_\alpha^{t-j_1}y_\beta^{-a_2}\xi_1=0.$$

Write

$$\hat{\xi}=y_\beta^{-a_1+a_2}y_\alpha^{-a_1-(t-j_1)}y_\alpha^{t-j_1}y_\beta^{-a_2}\xi_1.$$

Let

$$y_\alpha^{t-j_1}y_\beta^{-a_2}\xi_1=y_\beta^{-b_2}\xi_2 \quad\text{with}\quad \xi_2\in\underbrace{C_\alpha(C_\beta\dots}_{4\text{times}}(M))$$

of weight $\lambda_2=\lambda_1+(a_2-b_2)\beta-(t-j_1)\alpha$; the above is taken to be a minimal expression.

As $y_\alpha^{t-j_1}y_\beta^{-a_2}\xi_1=y_\beta^{-b_2}\xi_2$ is an x_β-invariant, we have

$$x_\beta\xi_2=0 \quad\text{and}\quad \lambda_2(H_\beta)+2b_2=b_2-1$$

But $\lambda_2(H_\beta)+2b_2=\lambda_1(H_\beta)+2a_2+(t-j_1)=a_2-1+(t-j_1)$. Thus

$$b_2=a_2+(t-j_1) \qquad \dots\text{(ii)}$$

So

$$\hat{\xi}=y_\beta^{-a_1+a_2}y_\alpha^{-a_1-(t-j_1)}y_\beta^{-a_2-(t-j_1)}\xi_2 \qquad \ldots \text{(I)}$$

Note also that $x_\alpha^t\xi_2=0$.

We now proceed with ξ_2 exactly in the same way as we did with ξ. We will see that the form required in the identity emerges by itself. Let $\xi_2=y_\alpha^{-b_3}\xi_3$, a minimal expression with $\xi_3\in C_\beta(C_\alpha(C_\beta(M)))$ of weight $\lambda_3=\lambda_2-b_3\alpha$.

Now by Theorem 2.6,

$$t\geqq b_3-\lambda_2(H_\alpha)=j_2\geqq 1 \quad \text{and} \quad x_\alpha^t y_\alpha^{t-j_2}\xi_3=0 \qquad \ldots \text{(iii)}$$

We have,

$$\lambda_2=\lambda_1-(t-j_1)\alpha-(t-j_1)\beta$$

so

$$\begin{aligned} j_2&=b_3-\lambda_1(H_\alpha)-(t-j_1)\\ &=b_3+a_1-3a_2+j_1-(t-j_1). \end{aligned}$$

Thus

$$b_3+(t-j_2)=3a_2-a_1+2(t-j_1) \qquad \ldots \text{(iv)}$$

Write

$$\begin{aligned} \xi_2&=y_\alpha^{-b_3-(t-j_2)}y_\alpha^{(t-j_2)}\xi_3\\ &=y_\alpha^{-3a_2+a_1-2(t-j_1)}y_\alpha^{t-j_2}\xi_3 \end{aligned} \qquad \ldots \text{(II)}$$

Let $y_\alpha^{t-j_2}\xi_3=y_\beta^{-b_4}\xi_4$, a minimal expression with

$$\xi_4\in C_\alpha(C_\beta(M)) \quad \text{of weight} \quad \lambda_4=\lambda_3-(t-j_2)\alpha-b_4\beta \qquad \ldots \text{(III)}$$

Just as in case of determining b_2, it can be proved that

$$b_4=2a_2-a_1+(t-j_1) \quad \text{and} \quad x_\alpha^t\xi_4=0 \qquad \ldots \text{(v)}$$

Let $\xi_4=y_\alpha^{-b_5}\hat{m}_1$, a minimal expression with $\hat{m}_1\in C_\beta(M)$. Once again, by Theorem 2.5,

$$t\geqq b_5-\lambda_4(H_\alpha)=j_3\geqq 1.$$

Using the expression for λ_4 in terms of λ_1 and the relations (i) to (v), we get:

$$j_3=b_5+2a_1-3a_2+j_1$$

Thus

$$-(b_5+(t-j_3))=2a_1-3a_2-(t-j_1).$$

So

$$\xi_4=y_\alpha^{-b_5-(t-j_3)}y_\alpha^{t-j_3}\hat{m}_1=y_\alpha^{2a_1-3a_2-(t-j_1)}\hat{m}_2 \qquad \ldots \text{(IV)}$$

with $\hat{m}_2\in C_\beta(M)$.

Using relations (I) to (IV), we get

$$\begin{aligned}\hat{\xi} &= y_\beta^{-a_1+a_2} y_\alpha^{-a_1-(t-j_1)} y_\beta^{-a_2-(t-j_1)} y_\alpha^{-3a_2+a_1-2(t-j_1)} \\ &\quad \cdot y_\beta^{-2a_2+a_1-(t-j_1)} y_\alpha^{-3a_2+2a_1-(t-j_1)} \hat{m}_2 \\ &= y_\beta^u y_\alpha^{3u+v} y_\beta^{2u+v} y_\alpha^{3u+2v} y_\beta^{u+v} y_\alpha^v \hat{m}_2,\end{aligned}$$

where

$$u = -a_1 + a_2 \quad \text{and} \quad v = 2a_1 - 3a_2 - (t - j_1) \qquad \ldots \text{(V)}$$

As mentioned earlier, we have, for any p, $q \in \mathbf{N} \cup \{0\}$ and $\eta \in C_\beta(M)$,

$$y_\beta^p y_\alpha^{3p+q} y_\beta^{2p+q} y_\alpha^{3p+2q} y_\beta^{p+q} y_\alpha^q \cdot \eta = y_\alpha^q y_\beta^{p+q} y_\alpha^{3p+2q} y_\beta^{2p+q} y_\alpha^{3p+q} y_\beta^p \cdot \eta.$$

However, we may not be able to use this directly in (V) as some of the exponents may be negative. A moment's thought will show that the above identity holds even if some of the exponents are negative. (The point is that the signs of $\{p, 3p+q, \ldots\}$ form connected blocks i.e. there can't be a positive sign inbetween two negative ones.) Thus

$$\hat{\xi} = y_\alpha^v y_\beta^{u+v} y_\alpha^{3u+2v} y_\beta^{2u+v} y_\alpha^{3u+v} y_\beta^u \hat{m}_2$$

and $\hat{m}_2$ being in $C_\beta(M)$ is of the form $y_\beta^{-r} m_2$, $m_2 \in M$. This shows that (**) is true in this case. A similar proof works when α is the longer root. This proves (**) completely in case of G_2.

As mentioned earlier, we omit the proofs of (**) in cases where J is of type A_2 or B_2. (Needless to say, these proofs are much simpler than the one given above.)

This completes the proof of Theorem 4.2.

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