KSOM ALGEBRA II 2021 MAY–AUG
NOTES AND EXERCISES: FINITELY GENERATED ABELIAN GROUPS

Splitting of Short Exact Sequences

Consider a short exact sequence of A-modules (where A is a ring with identity):

(1) $$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$$

(1) The following conditions on (1) are equivalent:
   (a) There is a homomorphism $j : M \to M'$ such that $j \circ i$ is the identity on M'.
   (b) There is a homomorphism $q : M'' \to M$ such that $p \circ q$ is the identity on M''.
   (c) There is a submodule N of M such that $M = iM' \oplus N$.

   The short exact sequence is split if the above conditions are satisfied.

(2) The sequence (1) splits if M'' is free.

(3) (Schanuel's Lemma) Let $0 \to K \to P \to M \to 0$ and $0 \to K' \to P' \to M \to 0$ be two short exact sequences, where P and P' are free. Then $K \oplus P' \simeq P \oplus K'$. (Put $X = \{(p, p') \in P \oplus P' \mid \pi(p) = \pi'(p')\}$, where π and π' denote respectively the surjections to M from P and P'. The projection (to the first co-ordinate) from $X \to P$ is surjective, and its kernel is isomorphic to K', so we have a short exact sequence $0 \to K' \to X \to P \to 0$, which moreover splits since P is free. Thus $X \simeq P \oplus K'$. A similar argument with the second projection shows that $X \simeq K \oplus P'$.)

Diagram Chasing

(1) (Snake Lemma)

(2) (Five Lemma) Consider the following commutative diagram of maps of modules:

Chain conditions: Noetherian and Artinian modules

[s:noethart]

(1) The following conditions are equivalent for a module M:
   - Every ascending chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots$ of submodules stabilises (that is, $M_j = M_{j+1}$ for all $j \gg 0$).
   - Every non-empty collection of submodules admits a maximal element (with respect to inclusion).

   A module satisfying these conditions is called Noetherian.

(2) A module M is Noetherian if and only if every submodule is finitely generated.

(3) The following conditions are equivalent for a module M:
   - Every descending chain $M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots$ of submodules stabilises (that is, $M_j = M_{j+1}$ for all $j \gg 0$).
   - Every non-empty collection of submodules admits a minimal element (with respect to inclusion).

   A module satisfying these conditions is called Artinian.

(4) Submodules and quotients of Noetherian (respectively, Artinian) modules are Noetherian (respectively, Artinian).

(5) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence. Then M is Noetherian (respectively, Artinian) if and only if M' and M'' are Noetherian (respectively, Artinian).

(6) A ring A is (left) Noetherian (respectively Artinian) if it is so as a (left) module over itself. A quotient of a Noetherian (respectively, Artinian) ring is Noetherian (respectively, Artinian).

(7) A finitely generated module over a Noetherian (respectively, Artinian) ring is Noetherian (respectively, Artinian).

Free modules and Finitely Generated Modules

Let A be any ring (with identity), not necessarily commutative, and let M be an A-module (left module).

(1) If M is finitely generated and it is free, then it is freely generated by a finite number of elements.

(2) Suppose that $A^r \simeq A^s$ with r finite. Then (a) s is finite; (b) if A is commutative, $r = s$.

Presentation of Modules

(1) Let $\varphi : A^m \to A^n$ be an A-linear map between free modules A^m and A^n. We can represent by a $m \times n$ matrix M with coefficients in A as follows.

(2) A module N is finitely presented if there is an exact sequence $A^m \to A^m \to M \to 0$ (here A^m and A^n are finitely generated free modules). Suppose that $\varphi : A^p \to M$ is a surjection from a finitely generated free module to a finitely presented module M. Then the kernel of φ is finitely generated. (Use Schanuel's Lemma.)

(3) A finitely generated module over a Noetherian ring is finitely presented.

Torsion in Finitely Generated Abelian Groups

(1) Let A be an integral domain (commutative ring having identity, with $1 \neq 0$ and no non-zero zero divisors). Let M be an A-module. The torsion submodule of M, denoted TM, is defined by $TM := \{m \in M \mid am = 0 \text{ for some } 0 \neq a \in A\}$. We say that M is torsion-free if $TM = 0$.

(a) The image of the torsion submodule under a module homomorphism lies in the torsion submodule.
(b) If $0 \to M' \to M \to M''$ is exact, then $0 \to TM' \to TM \to TM''$ is exact.
(c) M/TM is torsion-free. And $T(TM) = TM$.

(2) Suppose that a_1, a_2, ..., a_n are intgers that together generate the unit ideal. Then there is $n \times n$ with integer entries whose first row is $a_1, \ldots, a_n$ and which is invertible over the integers.

(3) A finitely generated torsion-free abelian group is free. (Let $x_1, \ldots, x_m$ generate a torsion-free abelian group M, with m being chosen to be the least possible. We claim that $x_1, \ldots, x_m$ are linearly independent (which is enough to show that M is free). Let $a_1x_1 + \ldots + a_mx_m = 0$ be a non trivial linear dependence relation among the x_1, ..., x_m. Using the fact that M is torsion-free, by factoring out the common factor in $a_1, \ldots,$ a_m, we may assume that the $a_1, \ldots, a_m$ generate the unit ideal. By item 2 above, there exists an invertible $m \times m$ integer matrix Z whose first row a_1, ..., a_m. Put $\underline{y} = Z\underline{x}$, where $\underline{x}$ is the $m \times 1$ matrix with entry x_i in row i. Letting y_i be the element in row i of $\underline{y}$, we see that $y_1 = 0$ on the one hand, and $y_1, \ldots, y_m$ generate M on the other. This contradicts the assumption of minimality of m.)

(4) Any subgroup of a finitely generated free abelian group is free.

(5) The additive group $\mathbb{Q}$ of rational numbers is torsion-free but not free. Indeed, any two rational numbers are linearly dependent over the integers.

(6) Let $\mathfrak{m}$ be the maximal ideal (x, y) in the polynomial ring $\mathbb{C}[x, y]$ in two variables over the complexes. Show that $\mathfrak{m}$ is torsion free but not free.

(7) An abelian group M is called torsion if it equals its torsion submodule, that is, if $M = TM$. A finitely generated abelian group which is torsion is finite.

Primary Decomposition of Finite Abelian Groups

Let N be a finite abelian group. Then the annihilator $\mathrm{Ann}\,(N) := \{a \in \mathbb{Z} \mid aN = 0\}$ of N is a non-zero ideal. Fix notation as follows:

- d is the positive integer such that $\mathrm{Ann}\,(N) = d\mathbb{Z}$
- $d = p_1^{r_1} \cdots p_k^{r_k}$ is the unique factorization of d as a product of primes.
- for $1 \leqslant i \leqslant k$, let e_i be an integer such that $e_i \equiv 1 \bmod p_i^{r_i}$ and $e_i \equiv 0 \bmod p_j^{r_j}$ for $j \neq i$. We have: $1 \equiv e_1 + \cdots + e_k \bmod d$, $e_ie_j \equiv 0 \bmod d$ (for $1 \leqslant i \neq j \leqslant k$), and $e_i^2 \equiv e_i \bmod d$ (for $1 \leqslant i \leqslant k$).
- $N_i := e_iN$
- For x an integer, let $(0 :_N x) := \{m \in N \mid xm = 0\}$.

Show the following:

(1) $N = N_1 \oplus \cdots \oplus N_k$.
(2) The stable value of $(0 :_N p_i) \subseteq (0 :_N p_i^2) \subseteq \ldots \subseteq (0 :_N p_i^\ell) \subseteq \ldots$ is reached at $(0 :_N p_i^{r_i})$ and equals N_i.

We call N_i the p_i-primary component of N. It is the unique Sylow p_i-subgroup of N.

Thus the primary decomposition of a finite abelian group N coincides with what we get by applying to N the structure theorem of finite nilpotent groups, namely, that they are direct products of their Sylow subgroups.

THE STRUCTURE OF PRIMARY COMPONENTS OF A FINITE ABELIAN GROUP

Let $A := \mathbb{Z}/p^k\mathbb{Z}$, where p is a prime.

(1) Let $\varphi : M \to N$ be an A-linear map between finitely generated A-modules M and N.
   (a) φ maps p^jM to p^jN, and so φ induces A-module maps $\varphi_j : p^{j-1}M/p^jM \to p^{j-1}N/p^jN$.
   (b) If the φ_j are isomorphisms for all $j \geqslant 1$, then so is φ.

(2) Let M be a finitely generated A-module. Let $s_j := \dim p^{j-1}M/p^jM$ (as a $\mathbb{Z}/p\mathbb{Z}$-vector space). Note that $s_j = 0$ for $j > k$ (since $p^{j-1}M = 0$ in that case).
   (a) $s_1 \geqslant s_2 \geqslant \dots$. Indeed, if $m_1, \dots, m_r$ are elements of M such that (the images of) $p^{j-1}m_1, \dots, p^{j-1}m_r$ in $p^{j-1}M/p^jM$ form a basis, then, for $i \leqslant j$, (the images of) $p^{i-1}m_1, \dots, p^{i-1}m_r$ in $p^{i-1}M/p^iM$ are linearly independent.
   (b) There exist elements $m_1, \dots, m_{s_1}$ in M such that, for every $j \geqslant 1$:
      (i) (the images of) $p^{j-1}m_1, \dots, p^{j-1}m_{s_j}$ in $p^{j-1}M/p^jM$ form a basis.
      (ii) $p^{j-1}m_{s_j+1} = \dots = p^{j-1}m_{s_1} = 0$.

(3) Put $t_k = s_k - s_{k+1} = s_k$, $t_{k-1} = s_{k-1} - s_k$, $\dots t_1 = s_1 - s_2$; and
$$P := \left(\frac{\mathbb{Z}}{p^k\mathbb{Z}}\right)^{t_k} \times \left(\frac{\mathbb{Z}}{p^{k-1}\mathbb{Z}}\right)^{t_{k-1}} \times \cdots \times \left(\frac{\mathbb{Z}}{p^1\mathbb{Z}}\right)^{t_1}$$
With M being as in item (2), we have an A-linear map $\varphi : P \to M$ defined as follows: the standard basis vector e_i maps to m_i for $1 \leqslant i \leqslant s_1$. This map is an isomorphism (using item (1) above).

SUMMARY: STRUCTURE THEOREM FOR FINITELY GENERATED ABELIAN GROUPS

We now summarise the results of the previous sections. Let M be a finitely generated abelian group. We let TM be the torsion submodule of M and consider the short exact sequence $0 \to TM \to M \to M/TM \to 0$. Since $\mathbb{Z}$ is a Noetherian ring, M is a Noetherian module, and in particular TM and M/TM are finitely generated.

Since M/TM is finitely generated and torsion-free, it is free (item (3) in the section on torsion in finitely generated modules). So the above short exact sequence splits (item (2) in the section on Splitting of Short Exact Sequences), which means (by item (1) of the same section) that $M = TM \oplus K$ where K is a free submodule of M isomorphic to M/TM. There is a unique non-negative integer r such that $M/TM \simeq \mathbb{Z}^r$ (see the section on Free Modules and Finitely Geneated Modules). We call r the <u>rank</u> of M. The torsion subgroup TM of M is finite (item (7) of the section on Torsion in Finitely Generated Abelian Groups).

By the results above on Primary Decomposition and the Structure of Primary Components, it follows that TM is uniquely the product of cyclic subgroups of prime power orders. From this we can deduce the existence of a unique sequence $d_1|d_2|\cdots|d_s$ of integers > 1 such that TM is the product of the cyclic groups of order $d_1, \ldots, d_s$.

Principal Ideal Rings

[s:pir]

Let A is commutative local ring with identity. (In particular $A \neq 0$.) Suppose that every ideal of A is principal. The following notation remains fixed:

- $\mathfrak{m}$ is the unique maximal ideal of A
- x an element of A such that $\mathfrak{m} = Ax$
- $I = \cap_{r\geqslant 1}\mathfrak{m}^r$
- M is an A-module
- $TM := \cup_{r\geqslant 0}(0 :_M x^r)$

(1) The items below explore properties of the ideals of A.
  (a) If $\mathfrak{p}$ is a prime ideal, then $\mathfrak{p}$ is either $\mathfrak{m}$ or 0.
  (b) If $\mathfrak{m}^r = \mathfrak{m}^{r+1}$, then $\mathfrak{m}^r = 0$.
  (c) An element a in A belongs to $\mathfrak{m}^r\setminus\mathfrak{m}^{r+1}$ (for some integer $r \geqslant 0$) if and only if $\mathfrak{m}^r \neq 0$ and $a = ux^r$ for some unit u in A.
  (d) Suppose that $\mathfrak{m}^r \neq 0$ for all positive integers r. Then I is a prime ideal.
  (e) $I = 0$.
  (f) Every non-zero ideal of A is of the form $\mathfrak{m}^r$ for some integer $r \geqslant 0$.