# AIS AT CMI JUNE–JULY 2022
# TUTORIAL SHEET 5: CONSEQUENCES OF WEDDERBURN AND MASCHKE

(1) Let $A$ be a finite dimensional semisimple algebra over an algebraically closed field $k$. Deduce the following as corollaries of Wedderburn's structure theorem:

(a) There are as many isomorphism classes of simple $A$-modules as the dimension of the centre of $A$ as a $k$-vector space.

(b) Let $V_1, \ldots, V_\ell$ be the complete list of pairwise non-isomorphic simple $A$-modules, and $d_1, \ldots, d_\ell$ their respective dimensions as $k$-vector spaces. Then $\dim_k A = d_1^2 + \cdots + d_\ell^2$. More precisely, we have

$$_A A = V_1^{\oplus d_1} \oplus \cdots \oplus V_k^{\oplus d_k}$$

where $_A A$ denotes $A$ as a left module over itself.

(c) (DENSITY) Given a list $V_1, \ldots, V_r$ of pairwise non-isomorphic simple $A$-modules and a list $\varphi_1, \ldots, \varphi_r$ of $k$-linear endomorphisms respectively of $V_1, \ldots, V_r$, there exists $a$ in $A$ such that, for each $i$, $1 \leqslant i \leqslant r$, the map $v \mapsto av$ on $V_i$ equals $\varphi_i$.

(d) (BURNSIDE'S LEMMA; special case of the previous item) If $V$ is a simple $A$-module, then the $k$-algebra homomorphism $A \to \mathrm{End}_k V$ (that defines $V$) is onto.

(2) Let $G$ be a finite group and $k$ be a field. A $k$-valued function $\varphi$ on $G$ is a class function if $\varphi$ is constant on conjugacy classes. The centre $\mathfrak{Z}(kG)$ admits the following description:

$$\mathfrak{Z}(kG) = \{\sum_{g \in G} \varphi(g) g \mid \varphi \text{ is a class function}\}$$

Thus the dimension of the centre of $kG$ equals the number of conjugacy classes in $G$.

(3) Let $G$ be a finite group and $k$ be a field. Suppose that characteristic of $k$ is either $0$ or, if positive, then does not divide $|G|$. Then the group algebra $kG$ is semisimple (Maschke's theorem). If, further, $k$ is algebraically closed, the conclusions in the first item above hold with $A = kG$.

## TUTORIAL SHEET 6: CHARACTER THEORY

(1) Let $V$ be a linear representation of a finite group $G$ over the complex numbers. Show that $V$ is unitarizable, that is, there exists an Hermitian $G$-invariant inner product on $V$. (Hint: Take any inner product and average it.)[1]

(2) Let $G$ be a group, $k$ a field, and $\rho : G \to GL_k(V)$ a $k$-linear representation of $G$ over $k$. Suppose that $d := \dim_k V < \infty$. By choosing a $k$-basis $\mathfrak{B}$ of $V$, we get a group homomorphism $\rho_{\mathfrak{B}} : G \to GL_d(\mathbb{R})$, which we may call the "matrix representation corresponding to $\rho$ with respect to the basis $\mathfrak{B}$". What is the matrix representation $\rho^*_{\mathfrak{B}^*}$ corresponding to the dual $\rho^* : G \to GL(V^*)$ with respect to the dual basis $\mathfrak{B}^*$ of $V$? (Hint: Answer: $(\rho^*_{\mathfrak{B}^*}(g))_{ij} = (\rho_{\mathfrak{B}}(g^{-1}))_{ij}$. In other words, it is the inverse-transpose of the matrix representation of $\rho$ with respect to $\mathfrak{B}$.)

(3) Deduce from the above two items that $\chi_{V^*} = \overline{\chi_V}$, for a finite dimensional complex linear representation $V$ of a group. Here $V^*$ denotes the dual of $V$ and $\chi$ the character.

(4) Let $G$ be a finite group and $\rho : G \to GL_n(\mathbb{R})$ a group homomorphism. Does there exist $\sigma$ in $GL_n(\mathbb{R})$ such that $\sigma\rho(g)\sigma^{-1}$ is an orthogonal matrix for every $g$ in $G$?

[1] This can be used to show that every complex representation $V$ of $G$ is semisimple. Given a sub-representation $W$, we could fix a $G$-invariant Hermitian inner product on $V$ and observe that $W^\perp$ (with respect to that inner product) is $G$-invaraint.