AIS AT CMI JUNE–JULY 2022
TUTORIAL SHEET 7: ISOTYPIC COMPONENTS AND PROJECTION FORMULAS

Notation: Throughout this sheet, G denotes a finite group, and φ a complex valued function on G. All linear representations are understood to be over the complex numbers, but they are not finite dimensional unless explicitly stated. The group homomorphism $G \to GL(U)$ defining a G-linear representation V is denoted $g \mapsto g_U$.

Linear representations of G are the same as modules for the group algebra $\mathbb{C}G$. This algebra is semisimple by Maschke. In particular, all linear representations of G are semisimple. Observe that irreducible linear representations are finite dimensional, being homomorphic images of the group algebra as a module over itself.

The element $\frac{1}{|G|}\sum_{g\in G}\overline{\varphi(g)}g$ of the group algebra is denoted by $\mathrm{Av}(\varphi)$. (Av stands for "average" and the reason for the complex conjugation in this definition will become clear in hindsight!) $\mathrm{Av}(\varphi)$ is central if and only if φ is a class function. Its image $\frac{1}{|G|}\sum_{g\in G}\overline{\varphi(g)}g_U$ in $\mathrm{End}_{\mathbb{C}}\, U$ for a G-representation U is denoted $\mathrm{Av}(\varphi, U)$.

(1) Let M be a semisimple module over some ring R. Then, by definition, M can be written as a direct sum of simple modules: $M = \bigoplus_\alpha V_\alpha^{\oplus m_\alpha}$, where V_α are the (pairwise non-isomorphic) simple R-modules, indexed by α (as it runs over some index set), and m_α the "multiplicity of V_α in M". Observe the following:

   (a) The submodule $V_\alpha^{\oplus m_\alpha}$ (for any fixed α) is independent of the chosen decomposition: for, by Schur's Lemma (first version), it is the sum of the images of all possible R-module homomorphisms from V_α to M. It is called the α-isotypic component of M or the V_α-isotypic component of M.

   (b) There is a unique R-module projection from M to any isotypic component, for such a projection must vanish on all the other isotypic components.

(2) (FIRST PROJECTION FORMULA, as Fulton-Harris calls it) Let U be a linear representation of G (not necessarily finite dimensional). Show that $\frac{1}{|G|}\sum_{g\in G} g_U$ (which is $\mathrm{Av}(1, U)$ where 1 denotes the constant function on G with complex value 1) is the unique G-projection to the subspace $U^G := \{u \in U \mid gu = g \ \forall\ g \in G\}$ of invariants. Note that the space U^G of invariants is nothing but the isotypic component corresponding to the trivial representation. (Proof: It is easily checked that the given operator on U commutes with the action of G and that it is a projection to U^G.)

(3) Suppose that φ is a class function. Let V be an irreducible representation. Note that V is necessarily finite dimensional.

   (a) By Schur's Lemma, $\mathrm{Av}(\varphi, V)$ is multiplication by a scalar. Let us denote this scalar by $\lambda(\varphi, V)$. In other words, we have the following equality (in the space of operators on V):
$$\mathrm{Av}(\varphi, V) \ := \ \frac{1}{|G|}\sum_{g\in G}\overline{\varphi(g)}g_V \ = \ \lambda(\varphi, V)\mathrm{Id}_V \tag{1}$$
   where Id_V the identity map on V.

   (b) To compute the scalar $\lambda(\varphi, V)$ (defined in the previous item), we can apply the trace functional to the operator equality (1)—this makes sense since V is finite dimensional—to obtain:
$$\lambda(\varphi, V)\cdot \dim V = \frac{1}{|G|}\sum_{g\in G}\overline{\varphi(g)}\chi_V(g) = \langle \varphi, \chi_V\rangle \tag{2}$$
   where, as usual, χ_V denotes the character of V.

(c) Putting $\varphi = \chi_W$ in (2) where χ_W is the character of an irreducible representation W (possibly isomorphic to V but possibly not), and invoking Schur's orthonormality of irreducible characters, we obtain:

$$\lambda(\chi_W, V) \cdot \dim V \;=\; \frac{1}{|G|}\sum_{g\in G}\overline{\chi_W(g)}\chi_V(g) \;=\; \langle \chi_W, \chi_V\rangle \;=\; \begin{cases} 1 & \text{if } W \simeq V \\ 0 & \text{if } W \not\simeq V \end{cases} \tag{3}$$

(d) (The Second Projection Formula, as Fulton-Harris calls it) On any linear representation U of G, not necessarily finite dimensional, the unique G-projection to the V-isotpyic component of U is given by:

$$\frac{\dim V}{|G|}\sum_{g\in G}\overline{\chi_V(g)}g_U \tag{4}$$

(Proof: Indeed, by (3) this operator acts as identity on V and kills all irreducible representations $W \not\simeq V$.)

(e) (Character theoretic proof that irreducible characters form a basis for class functions)[1] It is enough to show that there is no non-zero function that is orthogonal to the subspace spanned by the (irreducible) characters. Suppose that our class function φ is orthogonal to all irreducible characters. Then, by (1) and (2), the element $\mathrm{Av}(\varphi)$ of the group algebra kills all irreducible representations (and therefore all representations since all representations are semisimple by Maschke). In particular, it kills the group algebra itself (the left regular representation). But then, its action on the identity of the group algebra results in itself. Therefore $\mathrm{Av}(\varphi) = 0$, which means φ is identically zero.

(4) (Representation theoretic proof of the Orbit Counting Lemma) Let G be a finite group and X a finite set with a G-action. Consider the induced linear action of G on the free complex vector space $\mathbb{C}X$. Let us compute, in two different ways, the multiplicity of the trivial representation in $\mathbb{C}X$. On the one hand, the element $\sum_{x\in X} c_x x$ of $\mathbb{C}X$ is invariant under G if and only if the function c_x is constant on orbits of X, and hence the above multiplicity is the number of orbits of X. On the other hand, by Schur's orthonormaility of irreducible characters, this multiplicity is obtained as the inner product of the trivial character χ_{triv} with the character $\chi_{\mathbb{C}X}$ of $\mathbb{C}X$, which equals $\frac{1}{|G|}\sum_{g\in G}|X^g|$, where $X^g := \{x \in X \mid gx = x\}$ is the set of points in X fixed by g. Thus:

$$\text{the number of } G\text{-orbits in } X \;=\; \frac{1}{|G|}\sum_{g\in G}|X^g|$$

[1] Our proof in class that there are as many irreducible complex linear G-representations as conjugacy classes of G was based on Wedderburn's structure theorem. We will now show this using only character theory. Thus the basic theorems about the complex linear representations of a finite group can be developed without recourse to Wedderburn's structure theory of semisimple finite dimensional algebras.