AIS AT CMI JUNE–JULY 2022
TUTORIAL SHEET 4: SEMISIMPLICITY OF MODULES

(1) Any ring A with $1 \neq 0$ admits simple modules. (Hint: Use Zorn's lemma to show that there exist maximal left ideals in A. For such an ideal I, consider A/I as an A-module.)

(2) (Example of a non-semisimple finite dimensional complex algebra) Let B be the set of 2×2 upper triangular matrices with complex entries. With respect to standard matrix addition and multiplication, B is a $\mathbb{C}$-algebra. Let M be the set of 2×1 matrices with complex entries. Then with respect to multiplication from the left, M is a B-module. Show that M is not semisimple. (Hint: Consider the submodule N consisting of 2×1 matrices with the $(2, 1)$ entry being 0. Show that N has no complement. In fact, show that N is the only non-trivial proper submodule of M.) Conclude that B is not semisimple.

(3) True or false?: Submodules and quotient modules of semisimple modules are semisimple.

(4) True or false?: If N is a semisimple sub-module of a module M such that M/N is also semisimple, then M is semisimple.

(5) Suppose that a ring A is semisimple (which means, by definition, A as a left module over itself is semisimple). Then every A-module is semisimple.

(6) True or false?: Suppose that a ring A admits a faithful semisimple module. Then A is semisimple.

(7) Let A be a finite dimensional associative algebra over a field. Then the isomorphism classes of simple A-modules is finite. (Hint: For any ring A, any simple module is a quotient of A as a left module over itself: indeed if $0 \neq m$ in M then $a \mapsto am$ defines a module homomorphism—call it φ—from (the left module) A to M. Now let A be finite dimensional algebra. Fix a sequence $A = N_0 \supsetneq N_1 \supsetneq \ldots \supsetneq N_{k-1} \supsetneq N_k = 0$ of submodules such that N_i/N_{i+1} is simple for $0 \leqslant i < k$. Such a sequence exists: indeed the length k of such a sequence is bounded above by the dimension of A. Let r be the least integer such that $\varphi_{|N_r} = 0$. Then $r \geqslant 1$ and $M \simeq N_{r-1}/N_r$. Thus any simple module is isomorphic to one of the quotients N_{i-1}/N_i. $\square$)

(8) (Justification for the name "semisimple") Let V be a finite dimensional vector space over an algebraically closed field k and T a k-linear transformation of V. Then V becomes a module over the polynomial ring $k[t]$ by letting t act as T. The module V over $k[t]$ is semisimple if and only if T is semisimple (that is, diagonalisable).

(9) (Added by UK as a continuation of the first item above. The purpose is to give you practice using the basic isomorphism and correspondence theorems, the "nuts and bolts" of working with modules.) Fix a ring R. An R-module M is said to have the complement property if for every submodule N of M, there is a submodule P of M such that $M = N \oplus P$. Suppose that an R-module M has the complement property. Note that there is no finiteness assumption on R or M.
   (a) Show that any submodule of M has the complement property and so does any quotient of M.
   (b) Show that M has a simple submodule. (Hint: Start with a cyclic submodule of M.)
   (c) Show that M is a sum (and hence a direct sum) of simple submodules.
   (d) Show that a direct sum of simple submodules has the complement property. ("Direct" can be dropped.)

   Thus the three definitions given in class of "semisimple modules" for a finite dimensional module over an algebra remain equivalent over an arbitrary ring.