# AIS AT CMI JUNE–JULY 2022
# TUTORIAL 3A: FINITE DIMENSIONAL SIMPLE ALGEBRAS OVER A FIELD

Let $k$ be a field, $V$ a vector space of finite dimension $d$ over $k$, and $A := \operatorname{End}_k(V)$ the $k$-algebra of endomorphisms of $V$. We consider $V$ as a left module over $A$. The symbols ${}_AA$ and $A_A$ denote respectively $A$ as a left module and as a right module over itself.

Fixing a basis of $V$, we may identity $A$ and $V$ respectively with the matrix algebra $M_d(k)$ and the space of column matrices of size $d \times 1$ with entries in $k$, turning the action of $A$ on $V$ to be the usual matrix multiplication.

The dual $V^*$ has naturally a right $A$-module structure given as follows: for $f$ in $V^*$ and $a$ in $A$, let $(fa)(v) = f(av)$ for all $v$ in $V$. Continuing the theme of identifications of the previous paragraph, we may identity $V^*$ with the space of matrices of size $1 \times d$ over $k$. The pairing between $V^*$ and $V$ and the action of $A$ on $V$ (on the right) are all just the usual matrix multiplications.

(1) $A \simeq A^{\mathrm{opp}}$ by $\varphi \mapsto \varphi^{\mathrm{transpose}}$ (where $A^{\mathrm{opp}}$ denotes the opposite of $A$).
(2) $V$ is a simple module for $A$
(3) ${}_AA \simeq V^{\oplus d}$ (and so ${}_AA$ is semisimple)
(4) $V$ is the only simple module for $A$ and every finite dimensional module for $A$ is of the form $V^{\oplus e}$ for some non-negative integer $e$.
(5) $V^*$ is a simple right module and $A_A \simeq V^{*\oplus d}$ (and so $A_A$ is a semisimple right module).
(6) For a subspace $W$ of $V$, define

$$\ell_W := \{\varphi \in A \mid \varphi|_W = 0\} \text{ and } \rho_W := \{\varphi \in A \mid \varphi(V) \subseteq W\}$$

Then $\ell_W$ is a left ideal and $\rho_W$ a right ideal of $A$. Moreover, every left ideal of $A$ is of the form $\ell_W$ for some subspace $W$ and every right ideal of $A$ is of the form $\rho_W$ for some subspace $W$.
(7) The only two-sided ideals of $A$ are $0$ and itself. (An algebra is called simple if it admits precisely two two-sided ideals, namely $0$ and itself. Thus $A$ is simple, assuming $V \neq 0$.)

The following item outlines a proof of the structure theorem for finite dimensional simple algebras over algebraically closed fields. The only such algebras are endomorphisms of finite dimensional vector spaces as above. The proof uses a conclusion from Wedderburn's structure theorem for finite dimensional semisimple algebras over such fields.

(8) (a) (BURNSIDE'S LEMMA) Suppose that the field $k$ is algebraically closed and let $B$ be a $k$-algebra (not necessarily finite dimensional) that admits a simple finite dimensional module $W$. Then the $k$-algebra map $B \to \operatorname{End}_k W$ (defining $W$ as a $B$-module) is onto. (Proof: Let $S$ be the image of $B$ in $C := \operatorname{End}_k W$. By what has been said earlier on this tutorial sheet, we have ${}_CC \simeq W^{\oplus \dim W}$. Since $W$ is simple as an $S$-module, it follows that $C$ is semisimple as an $S$-module, and in turn that $S$ is a semisimple module over itself (since $S$ is an $S$-submodule of $C$). This means that $S$ is a finite dimensional semisimple algebra and, by a consequence of Wedderburn's struture theorem (see Tutorial sheet 5), it follows that $S \to C$ is onto. $\square$)

(b) (Structure theorem) Suppose that the field $k$ is algebraically closed and let $S$ be a finite dimensional simple $k$-algebra. Then $S \simeq \operatorname{End}_k W$ as $k$-algebras for some finite dimensional $k$-vector space $W$. (Proof: Let $W$ be a simple $S$-module. Such a module exists and is finite dimensional over $k$. The $k$-algebra map $S \to \operatorname{End}_k W$ (defining $W$ as an $S$-module) is surjective by Burnside's Lemma and injective because $S$ is simple. $\square$)

(9) In the assertions of the previous item, the hypothesis that $k$ is algebraically closed cannot be omitted. Indeed, let $k \subseteq K$ be any finite extension of fields. Then $K$ is a finite dimensional $k$-algebra. But there is no vector space $V$ over $k$ such that $K \simeq \operatorname{End}_k V$ as $k$-algebras (unless $K = k$).

Moreover, consider $K$ as a simple module over itself. The $k$-algebra map $K \to \operatorname{End}_k(K)$ is not surjective (unless $k = K$).

(10) Let $W$ be an infinite dimensional vector space over the field $k$. Then $\operatorname{End}_k W$ is not simple as a $k$-algebra: the set of finite rank endomorphisms forms a proper, non-zero, two-sided ideal.