AIS AT CMI JUNE–JULY 2022
TUTORIAL PROBLEM SHEET 2: ORBIT COUNTING LEMMA

(1) For each of the following groups G, determine the number of orbits for the action of G on the power set of G induced from the left regular action of G on itself: the cyclic group of order 10, the Dihedral group D_p where p is an odd prime, and the symmetric group $\mathfrak{S}_4$.

(2) Let H be a subgroup of a group G of finite index d. Then there exists a normal subgroup N of G that is contained in H and such that G/N imbeds in the symmetric group $\mathfrak{S}_d$. In particular, the index of N in G is at most $d!$. (Hint: Let $N = \cap_{g \in G} {}^g H$. Then the group homomorphism from G to $\mathfrak{S}_{G/H}$ defining the left action of G on G/H factors through G/N and the resulting map from G/N is injective.) In particular, an infinite simple group does not admit a finite index subgroup.

(3) Let X be a finite set consisting of at least 2 elements on which a group G acts transitively. Then there exists an element g of G that fixes no element of X. (Hint: We reduce to the case when G is finite using item (20) above. Invoke Burnside's lemma. The RHS of the lemma should work out to 1. Suppose that $|X^g| \geqslant 1$ for every g in G. Then the RHS is strictly bigger than 1, since $|X^{\mathrm{id}}| = |X| \geqslant 2$, a contradiction.) In particular:
- Given a finite but non-singleton conjugacy class in a group, there exists an element of the group that commutes with no element of the conjugacy class.
- Given a finite index proper subgroup H of a group, the conjugates of H cannot cover the group.

(4) Let G be a p-group and X a G-set. Then $|X| \equiv |X^G| \bmod p$. (Write X as a disjoint union of its orbits. The elements each of which forms a singleton orbit by itself are precisely those belonging to X^G. The cardinalities of the non-singleton orbits are divisible by p, since they divide $|G|$ and are bigger than 1.) Taking X to be G itself with the conjugation action, we conclude that every p-group has non-trivial centre. (Because X^G in that case is the centre, and X^G is non-empty since the identity element belongs to it.)

(5) A merry-go-round has 20 identically shaped wooden horses, equally spaced, on its circular periphery. The owner wants to paint 10 of the horses blue and 10 of them yellow. In how many different ways can this be done?

(6) In how many different ways that can the edges of a regular tetrahedron be painted with (at most) two colours?