# AIS AT CMI JUNE–JULY 2022
# TUTORIAL PROBLEM SHEET 1: GROUP ACTIONS ON SETS

(1) Let $H$ be a subgroup of a group $G$. For $g$ an element of $G$, let ${}^gH$ denote the conjugate $gHg^{-1}$ of the subgroup $H$.
- The $G$-sets $G/H$ and $G/{}^gH$ with their natural $G$-actions are isomorphic as $G$-sets. (The map $xH \mapsto xHg^{-1} = xg^{-1}{}^gH$ defines an isomorphism).
- Conversely, if, for a subgroup $H'$ of $G$, the $G$-sets $G/H$ and $G/H'$ are isomorphic (as $G$-sets) then $H' = {}^gH$ for some $g \in G$.

(2) The cardinality of any orbit under the action a finite group $G$ divides the order of $G$. In particular, the cardinality of any conjugacy class of $G$ divides the order of $G$.

(3) Let $\lambda = 1^{r_1}2^{r_2}\ldots$ be the cycle type of a permutation $\sigma$ in the symmetric group $\mathfrak{S}_n$ on $n$ letters. Let $z_\lambda := 1^{r_1}2^{r_2}\cdots r_1!r_2!\cdots$. Show that the number of conjugates of $\sigma$ in $\mathfrak{S}_n$ is $n!/z_\lambda$ and hence that the centraliser of $\sigma$ has cardinality $z_\lambda$.

(4) Let $\mathbb{F}$ be a finite field with $q$ elements. Let $V$ be a vector space of finite dimension $n$ over $\mathbb{F}$. Let $k$ be an integer $0 \leqslant k \leqslant n$. What is the number of $k$-dimensional subspaces in $V$?

(5) Let $\mathbb{F}$ be a field with exactly $7$ elements. Let $\mathfrak{M}$ be the set of all $2 \times 2$ matrices with entries in $\mathbb{F}$. How many elements in $\mathfrak{M}$ are similar to the following matrix?:

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(6) True of false?: Given an arbitrary subset $S$ of $G$, there is a unique maximal subgroup $K$ of $G$ such that $S$ is a union of right cosets of $K$.

(7) Let $G$ denote the cyclic group of order $m$. For $n$ a positive integer let $q(n)$ be the number of $G$-isomorphism classes of $G$-sets of cardinality $n$. Set $q(0) = 1$ (by definition). Show that the generating function $Q(t) := \sum_{n\geqslant 0} q(n)t^n$ equals

$$Q(t) = \frac{1}{\prod_{d|m}(1-t^d)}$$