Problems on basic representation theory of finite groups

based on exam problems in Upendra's courses

G denotes a finite group. G-module = a representation of G. All vector spaces are finite dimensional and, unless otherwise specified, over the field $\mathbb{C}$ of complex numbers.

1. Consider a G-module V and consider its dual V^*. Do the following independent problems concerning this situation. The ground field is arbitrary except as indicated.

a) Show that V is irreducible if and only if so is V^*.

b) Show that V and V^* are isomorphic G-modules if and only if V possesses a nondegenerate G-invariant bilinear form.

c) Suppose V possesses a nondegenerate G-invariant bilinear form. If V is irreducible, show that such a form is unique up to scalars, if the ground field is ... (fill in the blank). Prove the converse of this statement over a field of characteristic zero. [Optional: counterexample in positive characteristic?]

d) Assuming that V is an irreducible module with a nondegenerate G-invariant bilinear form, show that such a form must be symmetric or anti-symmetric, if the ground field is algebraically closed and has characteristic other than 2.

2. Show that the set of isomorphism classes of 1-dimensional representations of a group is an abelian group under tensor product. Use this to prove the following.

a) For an arbitrary G-module W and a 1-dimensional G-module L show that $W \otimes L$ is irreducible if and only if W is irreducible.

b) A finite abelian group G is isomorphic to the group G^* of its 1-dimensional complex representations (i.e., the group of its irreducible characters) and naturally isomorphic to G^{**}.

3. Show that given a G-module V over an arbitrary field k, the trivial representation k is a G-submodule as well as a factor G-module of $\mathrm{Hom}_k(V, V)$. Show that nonetheless k may not be a direct summand of $\mathrm{Hom}_k(V, V)$.

4. Part b below is the fundamantal fact about characters, from which most of the other basic facts about characters follow easily (how?). Prove part b by applying part a to the module $M = V^* \otimes W$.

a) Given a representation $\rho : G \to GL(M)$, show that the linear map $\phi = \frac{1}{|G|}\sum_{g\in G} \rho(g)$ is a G-invariant projection from M onto the subspace of fixed points $M^G = \{m \in M | \rho(g)m = m \ \forall g \in G\}$. What does the trace of ϕ calculate?

b) Prove that for G-modules V and W, $\dim \mathrm{Hom}_G(V, W) = (\chi_V, \chi_W) = (\chi_W, \chi_V)$.

c) Suppose you are given the character of a G-module V and no other information about G or V. In other words, you are given a list of $|G|$ complex numbers (possibly with repeats) and told that these are the values of $\chi_V(g)$ for some group G and some G-module V. Show with proof how can you decide whether V is an irreducible G-module.

5. a) Complete the given character table. You may not assume any information not explicitly provided. Can you guess the group?

b) Decompose $C \otimes E$ into a direct sum of irreducible representations.

c) Show that the given group has a unique normal subgroup and express it as a union of certain conjugacy classes. In general show how you can find all normal subgroups (expressed as unions of certain conjugacy classes) of a finite group G from its character table.

6. Calculations and short questions.

a) Consider the subgroup H of S_4 generated by the three cycle (123). Consider the one dimensional representation of H in which (123) acts by $e^{2\pi i/3}$. When this is induced to S_4, what is the dimesion of the resulting representation? Decompose it as a sum of irreducible representations of S_4. (Use the character table of S_4.)

b) Recall the example of a cyclic group G acting on a two dimensional *real* vector space V by rotations. V is an irreducible G-module but the complex representation $W = V \otimes_{\mathbb{R}} \mathbb{C}$ of G (obtained by extending scalars) is not irreducible (why?). Decompose W explicitly.

c) Find all groups that have exactly 2 nonisomorphic representations.

d) Given a positive integer n, is there a group with exactly n pairwise nonisomorphic representations?

e) Classify all irreducible complex representations of $\mathbb{Z}$. Are all finite dimensional representations of $\mathbb{Z}$ completely reducible?

f) For a group G, let M be a G-module in which each simple G-module appears with positive multiplicity. Show that M is faithful. Is the converse true?

g) For a simple G-module V, consider the linear map $\mathrm{Hom}_{\mathbb{C}}(V, V) \to \mathrm{Hom}_G(V, V)$ taking each matrix f to its average under the action of G on $\mathrm{Hom}_{\mathbb{C}}(V, V)$. What is the kernel of this averaging map?

7. For symmetric group enthusiasts. Proofs can be found in many places, for instance the the Fulton-Harris text and Jacobson's Basic Algebra II.

a) For a partition λ of a natural number d, recall the element $c_\lambda = a_\lambda b_\lambda$ inside $A =$ the complex group algebra of the symmetric group S_d. Prove that the S_d-modules $V_\lambda = Ac_\lambda$ form a complete set of pairwise non-isomorphic representations of S_d.

You may use the following facts (look up the proofs, which are of combinatorial flavor): for any $x \in A$, we have
(i) $c_\lambda x c_\lambda = n c_\lambda$ for some scalar n.
(ii) $c_\lambda x c_\mu = 0$ for any partition $\mu \neq \lambda$.

If ${c_\lambda}^2 = n_\lambda c_\lambda$, how is the scalar n_λ related to the irreducible S_d-module V_λ?

8. This presupposes knowledge of finite dimesnional irreducible representations of the Lie algebra $sl_2(\mathbb{C})$. For example see the texts by Humphreys and Fulton-Harris.

a) Recall the following: the three dimensional complex Lie algebra sl_2 has a basis X,Y and H and equations $[X, Y] = H$, $[H, X] = 2X$ and $[H, Y] = -2Y$. For each non-negative integer n there is a unique $n + 1$-dimensional irreducible representation V_n of sl_2. In V_n, up to scalars, there is a unique eigenvector v of H with eigenvalue n, which also satisfies $X.v = 0$. Show that $V_2 \otimes V_1$ decomposes as $V_3 \oplus V_1$ (find an explicit decomposition). Can you generalize to $V_k \otimes V_1$? To to $V_k \otimes V_\ell$?

b) For a representation M of sl_2 consider the linear operator $C = XY + YX + aH$, where a is a complex scalar. Find the value of a such that C is a Lie algebra endomorphism of M, i.e., the linear map C commutes with the action of X,Y and H. Calculate the action of C each finite dimensional irreducible representation of sl_2. A challenge: Can you use this to conclude that an arbitrary finite dimensional representation is completely reducible?

9. Show that over any field, a square matrix is similar to its transpose. Express this as a statement about any finite dimensional representation of an appropriate kind of group. The statement may not be as simple as one might hope, still the exercise is there to remind us that without linear (and multilinear) algebra, there is no representation theory.