TUTORIAL SHEET 9

Let p be a prime, G a p-group, k a field of characteristic p, and V an irreducible linear representation of G over k. Then V is trivial.

We outline several different proofs of this important fact, which also follows from the following basic theorem of Brauer to be proved later:

> *The number of isomorphism classes of irreducible representations of a finite group over an algebraically closed field of characteristic p equals the number of p-regular conjugacy classes of the group.*

(1) When k is finite, the triviality of V follows from Exercise 2 of Tutorial 4.

(2) Any endomorphism T of order $q = p^r$ of a vector space over k is unipotent: $(T-1)^q = T^q - 1 = 0$. Thus the image of G in $\mathrm{GL}(V)$ consists of unipotent elements. It now follows from Exercise 4 of Tutorial sheet 8 that V is trivial.

(3) Recall Clifford's observation (Exercise 6 of Tutorial sheet 1) that the restriction to a normal subgroup of a semisimple representation of an overgroup is semisimple. Proceed by induction on $|G|$. First assume $|G| > p$. Then there exists a non-trivial proper normal subgroup N of G (this follows for example from item (1e) of Tutorial 6). The restriction to N of V is semisimple. By induction, the action of N on V is trivial. In particular, the action of G on V goes down to G/N. But $|G/N| < |G|$, and so another application of the induction hypothesis shows that the action of G/N is trivial.

   The case $G = \mathbb{Z}/p\mathbb{Z}$ needs to be handled separately, but that is easily done: by commutativity, simple modules are 1-dimensional (by Schur's lemma; see item 1 of Tutorial 3), and the only root in k of $x^p - 1$ is 1.

(4) This proof is based on the class equation (item (1d) of Tutorial sheet 6). Let V be an irreducible representation over k of G. Let $0 \neq v$ be an element of V and consider the $\mathbb{Z}/p\mathbb{Z}$-span W of the orbit Gv of v. Then W is a finite G-set of cardinality a positive power of p. Since 0 is a G-fixed point in W, there is, by the class equation, at least one other G-fixed element in W. Denote by U the k-span of this element. Then U is a G-sub of V isomorphic to the trivial representation. Since V is irreducible, we must have $U = V$.