TUTORIAL SHEET 8

Commutant and bicommutant

Let A be a ring with identity, M an A-module. Let C and B be the commutant and bicommutant of M respectively.

(1) Take M to be A as a left module over itself. Identify C and B.

(2) Recall the following: let M be semisimple; then it is simple if and only if it is simple as a B-module. Can the assumption of semisimplicity be omitted?

(3) Let N be an A-direct summand of M. Let B_N denote the bicommutant of N. Since B preserves N, there is a ring homomorphism $B \to \operatorname{End}_{\mathbb{Z}} N$, which has image in B_N. Show that this need be neither surjective nor injective in general.

(4) Show that the bicommutant of $M^{\oplus n}$ as an A-module is the ring of homotheties as a B-module. (This can be paraphrased loosely as "bicommutants commute with self direct sums". Taking A to be a field k and M to be k itself as a left module, we recover the familiar fact that the centre of $n \times n$ matrices over k consists of scalar matrices.)

Extension of scalars

Let $k \subseteq L$ be fields and G a group. "Extending scalars from k to L," we may pass from linear representations over k to those over L.

(1) For finite dimensional representations V and W of G over k, we have $\operatorname{Hom}_G(V, W) \otimes_k L = \operatorname{Hom}_G(V \otimes_k L, W \otimes_k L)$.

(2) A linear representation is *absolutely irreducible* if it is irreducible under any extension of scalars (in particular, it is irreducible). A linear representation is *absolutely semisimple* if it is semisimple under any extension of scalars. The following are equivalent for a finite dimensional absolutely semisimple representation V over k:

For a finite group over a field of characteristic 0 any representation is absolutely semisimple.

   (a) V is absolutely irreducible;
   (b) $V \otimes \bar{k}$ is irreducible where $\bar{k}$ is an algebraic closure of k;
   (c) $\operatorname{End}_G(V)$ consists only of multiplications by elements of k.

Density theorem and Burnside's Lemma

(1) Let k be an algebraically closed field and A a k-algebra. Let $M_1, \ldots, M_n$ be simple A-modules, no two of which are isomorphic, and all of which are finite dimensional over k. Given $\phi_i \in \operatorname{End}_k M_i$, $1 \le i \le n$, there exists $a \in A$ such that the action of a on M_i is ϕ_i.

(2) (a) If V and W are finite dimensional irreducible representations over an algebraically closed field of groups G and H respectively, then $V \otimes W$ is an irreducible representation of $G \times H$. (Hint: The image of G in $\operatorname{End} V$ spans all of $\operatorname{End} V$, similarly for H and $\operatorname{End} W$; thus the image of $G \times H$ spans all of $\operatorname{End} V \otimes \operatorname{End} W = \operatorname{End}(V \otimes W)$, which implies the irreducibility of $V \otimes W$.)

(b) Conversely every finite dimensional irreducible representation of $G\times H$ arises uniquely thus. (Hint: Suppose that U is an irreducible module for $G\times H$. Let V be a simple G-sub of U, and W a simple H-sub of $\mathrm{Hom}_G(V,U)$. Then we get a non-trivial $G\times H$ homomorphism from $V\otimes W\to U$, which is an isomorphism (since U is simple by hypothesis and $V\otimes W$ is so by part (a)). The uniqueness follows since $V\otimes W$ is isotypic for G of type V and also for H of type W.)

(c) The proof of part (b) shows that, over any field, every irreducible representation of $G\times H$ arises as a quotient of $V\otimes W$ for some V irreducible over G and W irreducible over H.

(d) The statements in (a) and (b) fail over non-algebraically closed fields. Take, for example, G and H to be cyclic groups of orders 3 and 5 and the field to be consisting of the real numbers.

(3) Let V be a finite dimensional vector space over a field. A linear transformation u on V is *unipotent* if $u-1$ is nilpotent. The purpose of this exercise is to outline a proof of the following statement (which generalizes the assertion in Exercise 2 of Tutorial sheet 3):

> Given a subgroup of $\mathrm{GL}(V)$ consisting of unipotent elements, there exists a basis of V with respect to which all elements of the subgroup are represented by unipotent upper triangular matrices.

(Solution: Denote the subgroup by G. It is enough to show that G fixes some non-zero vector. We may assume without loss of generality that V is simple; further we may assume that the base field is algebraically closed. Fix g in G. Consider the equation $\mathrm{T}r(Xg)=\mathrm{T}r(X)$, where X is a variable in the space of all endomorphisms of V. Observe that it holds when X takes values in the group, and that it is linear in X. The linear span of elements of the group being all of $\mathrm{End}(V)$ by Burnside's lemma, it follows that the equation is an identity on $\mathrm{End}(V)$. But $(X,Y)\mapsto \mathrm{Tr}(XY)$ is a non-degenerate bilinear form on $\mathrm{End}(V)$, so g must be the identity, which means that the group is the trivial one.)

(4) Deduce from the density theorem Jacobson's original version of it: Let V be a simple module over a ring A. Then $D:=\mathrm{End}_A V$ is a division ring (by Schur's lemma). Consider V as a vector space over D. The action of A on V is *dense with respect to D*, i.e., given finitely many D-linearly independent elements $v_1, \ldots, v_m$ of V and an equal number of arbitrary elements $w_1, \ldots, w_m$ of V there exists a in A such that $av_1=w_1, \ldots, av_m=w_m$.

(5) Let V be a finite dimensional vector space over a division ring D. Let A be a subring of D-endomorphisms of V. Assume that A is 2-*transitive*, i.e., given any two linearly independent elements v, w of V and any two elements v', w' of V, there exists a in A such that $av=v'$ and $aw=w'$.[1] Show that the commutant of A is D and that $A=\mathrm{End}_D V$.

[1]This should be taken to mean that A is also 1-transitive (to cover for the situation when there may not exist two linearly independent elements, lest the hypothesis on A become vacuous).