TUTORIAL SHEET 6

In what follows, p denotes a prime number, and $q = p^r$ for some integer $r \geq 1$. A conjugacy class in a finite group is *p-regular* if the order of any of its elements is prime to p; is *p-singular* otherwise. Exercises 2–4 and the last item in Exercise 1 are purely group theoretic statements. These will be used later.

(1) Listed below is a series of elementary but important observations. Let G be a finite group acting on a finite set X.
   (a) The G-orbits of X form a partition of X (evidently).
   (b) In particular, $|X| = |X^G| + \sum |\text{orbits}|$, where X^G is the subset of X consisting of the G-fixed points of X and the sum is over the non-singleton orbits.
   (c) Taking X to be G acted upon by itself by conjugation we get the CLASS EQUATION: $|G| = |\text{Centre of } G| + \sum |\text{class}|$, where the sum is taken over the non-singleton conjugacy classes.
   (d) When G is a p-group, we get $|X| \equiv |X^G| \bmod p$, since the non-trivial orbits have cardinalities divisible by p.
   (e) The center of a p-group is non-trivial. (Hint: Combine the previous two items.) If the group has order p^2, it is abelian.

(2) The number of p-regular conjugacy classes in $\mathrm{SL}(2, q)$ is q.

(3) For g an element of finite order in a group, there is a unique expression[1] $g = su$, with s, u in the group, such that
   - the order of s is coprime to p, that of u is a power of p;
   - s and u commute.

   Evidently, s has order r and u order p^e, where the order of g is written as $p^e s$ with $(p, s) = 1$.

(4) Let elements x, y of a group be non-conjugate. Let their orders be coprime to p. Then x^{p^e} and y^{p^e} are non-conjugate (for all $e \geq 0$).

[1]This is the Jordan decomposition when the finite group G is realized as a linear algebraic group over a (perfect) field of characteristic p.