TUTORIAL SHEET 5

Artinian and Noetherian modules

Exercises 1–5 may not require any writing at all! Exercises 6–8 are more interesting.

(1) A module is Artinian (respectively Noetherian) if and only if it is so over its ring of homotheties[1].

(2) An infinite direct sum of non-zero modules is neither Artinian nor Noetherian. A vector space is Artinian (respectively Noetherian) if and only if its dimension is finite.

(3) The following would continue to be true if we replaced 'Artinian' by 'Noetherian':
- Submodules and quotient modules of Artinian modules are Artinian.
- If a submodule N of a module M and the quotient M/N by it are Artinian, then so is M.
- A finite direct sum is Artinian if and only if each of the summands is so.

(4) A module is both Artinian and Noetherian if and only if it has finite length.

(5) Any subset S of a Noetherian module contains a finite subset that generates the same submodule as S.

(6) A surjective endomorphism of a Noetherian module is bijective. (Hint: Given such an endomorphism u, consider $\operatorname{Ker} u \subseteq \operatorname{Ker} u^2 \subseteq \ldots$. This stabilises, so there exists n such that $\operatorname{Ker} u^n = \operatorname{Ker} u^{n+1}$. Observe also that u^m is surjective for all $m \geq 1$. If $ux = 0$, choose y such that $x = u^n y$. Then we have $0 = ux = u(u^n y) = u^{n+1}x$, so that $y = u^n x = 0$, and so $x = u^n y = 0$.)

(7) An injective endomorphism of an Artinian module is bijective. (Hint: See the hint to the previous exercise. Modify appropriately.)

(8) (Fitting) Let M be of finite length (equivalently, both Noetherian and Artinian). Let u be an endomorphism of M that is neither nilpotent nor invertible. Then for sufficiently large n we have a direct sum decomposition

$$M = \operatorname{Ker} u^n \oplus \operatorname{Im} u^n$$

[1]Given a module M over a ring A, we have the ring homomorphism $A \to \operatorname{End}_{\mathbb{Z}} M$ defining the A-module structure on M. The image of this homomorphism is called the *ring of homotheties* and denoted sometimes by A_M.

Artinian and Noetherian rings

Exercises 1–4 may not need any writing at all!

(1) If A contains a subring (with unity) which is a division ring, and if A is finite dimensional as a left module over the division ring, then A is Artinian and Noetherian.

(2) A principal ideal ring is Noetherian.

(3) It is a theorem (due to Hopkins, which we might see later on) that Artinian rings are Noetherian. So, if a ring is Artinian, then ${}_AA$ has finite length, called the *left length* (just *length* if A is commutative).

Artinian rings are Noetherian.

(4) The following properties are elementary to prove. They are stated for Artinian rings but the corresponding statements hold also for Noetherian rings.
- A finite direct product of Artinian rings is Artinian.
- A quotient of an Artinian ring (by a two sided ideal) is Artinian.
- A finitely generated module over an Artinian ring is Artinian.

(5) The following statement is made only for Noetherian rings (why not for Artinian rings?): A commutative ring that admits a faithful Noetherian module is Noetherian.

(6) Let a, b be elements of a (left) Noetherian ring A. If $ab = 1$, then $ba = 1$.

(7) An artinian subring of a division ring is a division ring.