TUTORIAL SHEET 20
RADICAL SERIES, SOCLE SERIES, PIM'S, ETC.

Let k be an algebraically closed field of positive characteristic, A a finite dimensional algebra over k, and M a finitely generated module over A. The symbols $\mathfrak{Rad}$ and $\mathfrak{soc}$ are used to denote radical and socle.

(1) Find the radical series and socle series of the algebra of upper triangular matrices with entries in k.

(2) The radical length and socle length of M are equal (and called the *Loewy length*). Denoting by ℓ the Loewy length of M, we have $\mathfrak{Rad}^i M \subseteq \mathfrak{soc}^{\ell-i} M$.

(3) Let P be a finitely generated projective A-module. Set $M = P/\mathfrak{Rad}\, P$. The following are equivalent:
- M is simple.
- M is indecomposable.
- P is indecomposable.

(4) Let S be a simple A-module and P the corresponding PIM. Show that $\dim_k \operatorname{Hom}_A(P, M)$ is the multiplicity of S in a composition series for M.

(5) Recall the result that $P/\mathfrak{Rad}\, P \simeq \mathfrak{soc}\, P$ for a (finitely generated) projective module P over the group ring over k of a finite group G. Show by example that this is not true for arbitrary A.

(6) Prove or disprove: any quotient of an indecomposable module is indecomposable.