TUTORIAL SHEET 2

(1) Give examples of indecomposable non-simple modules over $\mathbb{Z}$, over kG (the group ring of a finite group G over a field k), and A a finite dimensional algebra over a field k. (In the case of the group ring, the field has to be of finite characteristic; over fields of characteristic zero, there would be no example.)

(2) "Identify" the group rings over fields $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ of cyclic groups, of the permutation group $\mathcal{S}_3$, and of the group Q_8 consisting of the quarternions $\{\pm 1, \pm i, \pm j, \pm k\}$.

(3) What are all the simple and semisimple modules over $\mathbb{Z}$, over $k[X]$?

(4) What are all the indecomposable finitely generated modules over $\mathbb{Z}$?

(5) The "defining representation" V of $\mathrm{GL}_k(V)$ is simple. The "defining representation" of $M_n(k)$ consisting of $n \times 1$ matrices is simple.

(6) Given a linear transformation T of a finite dimensional vector space V over a field k, we can think of V naturally as a $k[X]$ module (by letting X act as T). Determine when this module is simple, semisimple, indecomposable, cyclic.

More questions from Tutorial 1

(1) The linearization of a permutation representation always admits a trivial sub-representation.

(2) The set $\mathrm{Inn}(G)$ of inner automorphisms forms a normal subgroup of the group $\mathrm{Aut}(G)$ of all autormorphisms. The action of $\mathrm{Aut}(G)$ on isomorphism classes of irreducible representations of G goes down to $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$.

(3) Show that $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$ are conjugate in $\mathrm{GL}_2(\mathbb{R})$ but not in $\mathrm{SL}_2(\mathbb{R})$ (for $t \neq 0$).

(4) Prove that the "transpose inverse" map is the only non-trivial element of $\mathrm{Out}(\mathrm{GL}_n(\mathbb{R}))$.

(5) Show that the matrix representation $x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ (where p is a prime) as 2×2 matrices over the field $\mathbb{Z}/p\mathbb{Z}$ has exactly one proper sub-representation.