TUTORIAL SHEET 19

COMPLEX IRREPS OF SUPER-SOLVABLE GROUPS ARE MONOMIAL

A finite dimensional representation of a group G is called *monomial* if with respect to some basis the matrices representing elements of the group are all monomial.[1] A representation induced from a one-dimensional representation of a subgroup is monomial.

A group G is *super-solvable* if there exists a sequence of subgroups

(*) $\{1\} = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_{n-1} \subseteq G_n = G$

such that (1) the subgroups G_i are all normal *in* G; and (2) the quotients G_{i+1}/G_i are all cyclic (for $0 \leq i \leq n-1$). The dihedral groups D_n are super-solvable but not nilpotent.[2] Replacing 'cyclic' by 'abelian' in the second condition above gives us the definition of a solvable group.

Our goal in this tutorial is to give the proof in Serre's book of the following: [t:supsolm]

Theorem 1. *Complex irreps of finite super-solvable groups are monomial.*[3]

The following technical proposition, which enters the proof, is of independent interest, so we have given it a name. [p:clifford2]

Proposition 2. (Clifford 2) *Let V be a finite dimensional irrep of a group G over a field k. Let N be a normal subgroup of G and $V|_N$ the restriction of V to N. Then one or the other of the following two conditions holds:*

(a) *V is induced from a proper subgroup H of G containing N.*
(b) *$V|_N$ is isotypic.*

Proof. We know that $V|_N$ is semisimple (Clifford 1). For an N-sub U of V:

(1) $\sum_{g \in G} {}^gU = V$ since V is simple;
(2) if U is isotypic, so is ${}^gU$ for any g in G.

Thus G acts transitively on the isotypic components U_j, where $V|_N = U_1 \oplus \cdots \oplus U_r$ is the isotypic decomposition of $V|_N$.

Suppose that (b) does not hold. Then there is more than one U_i. Let $H := \{g \in G \mid {}^gU_1 = U_1\}$. Observe that H is proper and that $\mathrm{Ind}_H^G U_1 = V$. □

Let us now prove Theorem 1. Proceeding by induction on the order of the group, we may assume that the irrep is faithful. If the group is abelian, all irreps are 1-dimensional, so monomial, and we are done. So assume not. Let A be a non-central abelian normal subgroup of G: to see that such an A exists, consider the sequence (*) and let m be maximal such that G_m is not central; then G_{m+1} has the desired property (because 'cyclic modulo central is abelian'). Consider $V|_A$. If it is isotypic, that is, condition (b) in Proposition 2 holds, then each element of A acts like a scalar, so A belongs to the centre (by faithfulness), a contradiction. Thus (a) holds and we are done by induction.

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[1] A matrix is monomial if in every row and every column it has precisely one non-zero entry.

[2] We recall what it means for a group G to be nilpotent: set $\mathfrak{C}^0G := G$, $\mathfrak{C}^1G := (G,G)$, ..., $\mathfrak{C}^{i+1}G := (G, \mathfrak{C}^iG)$, ... We call G *nilpotent* if $\mathfrak{C}^nG = \{1\}$ for some n. A p-group is nilpotent. (Proof: induction on order; p-groups have non-trivial centre (by class equation); quotient by centre is nilpotent by induction; central extension of nilpotent is nilpotent.)

[3] The proof works over an algebraically closed field of any characteristic.