TUTORIAL SHEET 17

BRAUER'S THEOREM ON NUMBER OF SIMPLE kG-MODULES

Let p be a prime and $q = p^r$ for $r \geq 1$ an integer. Let k be a field, algebraically closed of characteristic p. Let G be a finite group and kG the group ring of G with coefficients in k. Let V denote the defining representation of $\mathrm{SL}(2,k)$ and V_n denote the symmetric n^{th} power of V.

(1) Show directly, without using Brauer's theorem, that if G is cyclic of order $n = p^e r$ where $(r,p) = 1$, then there are exactly r inequivalent simple kG-modules.
(2) Show that V_p is not a simple module for $\mathrm{SL}(2,p)$. (In fact, it is not a simple module even for $\mathrm{SL}(2,k)$, and so not for $\mathrm{SL}(2,q)$ no matter what q is.)
(3) List the dimensions of the simple kG-modules for $G = \mathrm{SL}(2,q)$ and $q = 5^2$.
(4) Let $p = 3$ and $G = \mathrm{SL}(2,p)$. Find a composition series for V_3.
(5) Let $T = [kG, kG]$ (as in the proof of Brauer's theorem). Show the following:
   (a) $(x+y)^p \equiv x^p + y^p \bmod T$ for x and y in kG;
   (b) $x^p \in T$ for $x \in T$;
   (c) $(x+y)^{p^e} \equiv x^{p^e} + y^{p^e} \bmod T$ for x and y in kG.