TUTORIAL SHEET 16
COMPLETIONS

Notation: A - a ring, I - a two-sided ideal of A, V - a left module over A, d_I-metric induced by I.

(1) Let $A = \mathbb{Z}$ and $I = p\mathbb{Z}$ (p-prime). Show that $(\mathbb{Z}, d_I)$ is not complete.

(2) Give an example of A and I such that (A, d_I) is complete.

(3) If for some integer $n \geq 0$ we have $I^n V = \{0\}$ then the metric space (V, d_I) is complete.

(4) Find examples of commutative rings R and proper ideals I such that
$$\bigcap_{n=0}^{\infty} I^n \neq \{0\}.$$

(5) Let d_I be defined on A-modules A, V and W. Then the following functions are continuous:
   (a) The addition law $+ : V \times V \to V$ on V.
   (b) The module action $A \times V \to V$.
   (c) The ring operations on A.
   (d) Any $f \in \mathrm{Hom}_A(V, W)$.