TUTORIAL SHEET 14

LECTURE BY PROFESSOR BHASKAR BAGCHI OF 08 JUNE 2010

Let G be a finite group. We fix an algebraically closed field of characteristic 0. All representations and characters are implicitly taken to be over this field. Let

- $\chi_1, \ldots, \chi_r$ denote the irreducible characters
- $C_1, \ldots, C_r$ denote the conjugacy classes
- C_j also stand for the sum of all the elements of the class C_j
- x_j denote a representative element of the class C_j
- c_j denote the cardinality of the class C_j
- $\omega_{ij} := c_j\chi_i(x_j)/\chi_i(1)$, the "degree class-order normalized character values"

Character values of symmetric groups are integers

(1) Let e be an exponent of G. Suppose that, for all m such that $(m, e) = 1$, and all $x \in G$, x and x^m are conjugate. Then the character values are ordinary integers.
Solution: Let χ be an irreducible character and $x \in G$. Let ζ be a primitive e^{th} root of unity. Then we can write $\chi(x) = \zeta^{n_1} + \cdots + \zeta^{n_r}$, for $\chi(1) = r$, and for some $1 \leq n_1, \ldots, n_r \leq e$. For any integer m, we have $\chi(x^m) = \zeta^{n_1 m} + \cdots + \zeta^{n_r m}$.
We claim that $\chi(x)$ is fixed by all elements of the Galois group of $\mathbb{Q}[\zeta]$ over $\mathbb{Q}$. Let φ be such an element. Then $\varphi(\zeta) = \zeta^m$ for some m such that $(m, e) = 1$, so that $\chi(x) = \zeta^{n_1 m} + \cdots + \zeta^{n_r m}$, which as observed above is $\chi(x^m)$. But $\chi(x) = \chi(x^m)$ by the hypothesis on the group. Thus the claim is proved.
It follows from the claim that $\chi(x)$ is rational. Since it is an algebraic integer (being a sum of algebraic integers), it follows that it is an integer. $\square$

(2) The symmetric group satisfies the hypothesis of the previous exercise. So all entries of its character table are ordinary integers.

The ω_{ij} are algebraic integers

(1) Let A be a principal ideal domain and K its field of fractions. Then every finite dimensional representation V of G over K can be realized over A. What this means is that there is an A-free module with an A-linear action of G on it whose extension of scalars to K is isomorphic to V.
Solution: Consider the A-span V_0 of a K-basis for V. The A-module $\sum_{g\in G} gV_0$ has the requisite properties. $\square$

(2) Show that C_j acts as the scalar ω_{ij} on the irreducible representation with character χ_i.
Solution: Since C_j belongs to the centre, it acts like a scalar, the value of which clearly is $\chi_i(C_j)/\chi_i(1) = c_j\chi_i(x_j)/\chi_i(1) = \omega_{ij}$. $\square$

(3) The ω_{ij} are algebraic integers.
Solution: Any representation can be realized over the algebraic closure of $\mathbb{Q}$ in our fixed base field (why?). It may further be realized (by adjoining matrix elements) over a finite extension F of $\mathbb{Q}$. Now, by an earlier exercise, it may be realized over any localization at a prime of the ring A of algebraic integers in F (since such localizations are principal ideal domains with fraction field F). This means that

the ω_{ij} belong to all such localizations. But the intersection of these localizations being A, we are done. $\square$

(4) (Contributed by C. P. Anil Kumar) Apparently, Frobenius's original proof that the ω_{ij} are algebraic integers was along the following lines: consider the centre $\mathbb{Z}[C_1, \ldots, C_r]$ of the integral group ring $\mathbb{Z}G$. This being a finite module over $\mathbb{Z}$, the C_j satisfy monic equations with integral coefficients. Since any equation an operator satisfies is also satisfied by its eigenvalues, and C_j acts on the irreducible representation corresponding to χ_i as the scalar ω_{ij} (see earlier exercise), it follows that ω_{ij} satisfies the equation that C_j does. (Serre's book has a more general version of this argument.)

The degrees of irreps divide the order of the group

The degree $\chi_i(1)$ of any irreducible character χ_i divides $|G|$.

Solution: The left hand side in the following being an algebraic integer and the right hand side a rational, the result follows:

$$\sum_j \omega_{ij}\overline{\chi_i(x_j)} = (1/\chi_i(1))\sum_j c_j\chi_i(x_j)\overline{\chi_i(x_j)} = |G|/\chi_i(1) \qquad \square$$

On characters of non-abelian simple groups

(1) For any character χ and any x in G, we have $|\chi(x)| \leq \chi(1)$. Equality holds if and only if x acts as a scalar on the corresponding representation.

(2) The group G is non-abelian simple if and only if $|\chi(x)| < \chi(1)$ for all $x \neq 1$ in G and all irreducible characters $\chi \neq 1$.

Solution: First suppose that G is not simple. Then it has a non-trivial normal subgroup, say N. Let χ be a non-trivial irreducible character of G/N pulled-back to G. For all $x \in N$, we have $\chi(x) = \chi(1)$.

Now suppose that G is abelian simple. Then $|\chi(x)| = \chi(1) = 1$ for all x in G and all irreducible characters χ.

Conversely, suppose that $|\chi(x)| = \chi(1)$ for some $x \neq 1$ and $\chi \neq 1$. Consider the representation V corresponding to χ. By an exercise in the previous section, x acts like a scalar on V. If $G \to \mathrm{GL}(V)$ has a kernel, then of course G is not simple. If not, then G is isomorphic to its image in $\mathrm{GL}(V)$, so x belongs to the centre of G. $\square$

(3) Let α be the arithmetic mean of finitely many roots of unity. Assume that α is an algebraic integer. Then either $|\alpha| = 1$ or $\alpha = 0$.

Solution: The hypotheses on α hold also for all of its Galois conjugates. Consider the minimal polynomial for α over $\mathbb{Q}$. This has integral coefficients and its constant term c is the product of all conjugates of α. The first hypothesis on α implies that $|\beta| \leq 1$ for all conjugates $|\beta|$ of α. Thus $|c| \leq 1$. If $c = 0$ then $\alpha = 0$. Otherwise $|c| = 1$, which means that α and all its conjugates have modulus 1. $\square$

(4) Let G be non-abelian simple. If $(\chi_i(1), c_j) = 1$ for an irreducible character $\chi_i \neq 1$ and a conjugacy class $C_j \neq \{1\}$, then χ_i vanishes on C_j.

Solution: Let a and b be integers such that $a\chi_i(1) + bc_j = 1$. Multiplying by $\chi_i(x_j)/\chi_i(1)$ gives $a\chi_i(x_j) + b\omega_{ij} = \chi_i(x_j)/\chi_i(1)$. The left hand side is an algebraic integer and the right hand side is an arithmetic mean of roots of unity. By the previous exercise, it follows that if $\chi_i(x_j) \neq 0$, then $|\chi_i(x_j)| = \chi_i(1)$. But this is a contradiction to the assertion of Exercise (2) above. $\square$

Two theorems of Burnside

(1) (Burnside) A simple group has no conjugacy class of prime power order > 1.
Solution: We may assume the group to be non-abelian. Let $C_j \neq \{1\}$ be a conjugacy class of prime power order p^r. By an exercise in the previous section, χ_i vanishes on C_j unless p divides $\chi_i(1)$ or $\chi_i = 1$. Combining this with the orthogonality relation $\sum_i \chi_i(x_j)\chi_i(1) = 0$ we get $-1/p = \sum_i \chi(x_j)(\chi_i(1)/p)$, where the sum is taken over i such that p divides $\chi_i(1)$. But character values being algebraic integers, the right hand side is an algebraic integer. But not so the left hand side, leading to a contradiction. □

(2) (Burnside) The order of a non-abelian simple group has at least three distinct prime factors.[1]
Solution: Let the order of the group be of the form $p^a q^b$, where p and q are prime. It suffices, by the previous exercise, to show that there is a conjugacy class of prime power order. Let P be a Sylow p-subgroup ($a \geq 1$ without loss of generality). Choose $x \neq 1$ in the centre of P (see item (1e) of Tutorial 6). The conjugacy class of x has order a power of q. □

Miscellaneous

(1) Conjecture (Parker): The permanent of the character table of the symmetric group on n letters vanishes for $n \geq 2$.

(2) Find a formula for the square of the absolute value of the determinant of the character table of a finite group.
Solution: Let A be the matrix of the character table. The orthogonality relations amount to saying that A is orthogonal when the entries in the column corresponding to χ_j are multiplied by $\sqrt{c_j/|G|}$. The determinant of an orthogonal matrix being ± 1, we get

$$|\det A| \cdot (\prod_j (\sqrt{c_j/|G|}) = 1 \quad \text{and so} \qquad |\det A|^2 = \prod_j |G|/c_j.$$

(3) (A challenge problem) The symmetric group $\mathcal{S}_n$ carries a natural metric: $d(x,y)$ is the number of symbols moved by $x^{-1}y$. The group $\mathcal{S}_n \times \mathcal{S}_n$ acts on this metric space as isometries. Can you prove the following result of Kiyoto using rudiments of representation theory?:

> If G is a subgroup of $\mathcal{S}_n$ and L a finite subset of positive integers that contains $\{d(x,y) \mid x \text{ and } y \text{ in } G,\ x \neq y\}$, then the order of G divides the product of the elements in L.

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[1] There are simple groups whose orders have only three distinct prime factors, e.g., A_5 which has order 60, or $\mathrm{PSL}_2(\mathbb{F}_7)$ which has order 168.

Suppose that the order of a group is of the form $p^a q^b$ where p and q are primes. Then, by the theorem, it has a non-trivial normal subgroup (except in the case it is cyclic of prime order). Since the hypothesis on the order descends to factors and subgroups, we conclude that groups of such order are solvable.