TUTORIAL SHEET 13

Let p be a prime and $\mathbb{F}_q$ the finite field with $q = p^r$ elements. We outline a proof that there are $q^{n-1}(q-1)$ p-regular conjugacy classes in the group $G = \mathrm{GL}_n(\mathbb{F}_q)$ of invertible $n \times n$ matrices with entries in $\mathbb{F}_q$.

The relevance for representation theory of the above is that as a consequence there are $q^{n-1}(q-1)$ isomorphism classes of irreducible representations of $\mathrm{GL}_n(\mathbb{F}_q)$ over an algebraically closed field of characteristic p.

1. p-regularity is semisimplicity

Let $g = su$ be the Jordan decomposition (Exercise 3 of Tutorial 6) of an element g of G. Evidently the class of g is p-regular if and only if $u = 1$. Since s has order prime to p, it is semisimple (i.e., it is diagonalizable over the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$). Thus p-regular elements are semisimple.

Conversely semisimple elements are p-regular. Indeed, any invertible diagonal matrix over $\overline{\mathbb{F}}_q$ has order prime to p (for every non-zero element of $\overline{\mathbb{F}}_q$ has order prime to p).

Thus p-regular conjugacy classes are the same as semisimple conjugacy classes.

2. Semisimple classes and characteristic polynomials

For $g \in G$, let C_g denote its characteristic polynomial. Each C_g has the following properties:

- it is monic of degree n;
- it has coefficients in $\mathbb{F}_q$;
- the constant term is non-zero.

Let $\mathfrak{C}$ be the set of polynomials with these properties. The cardinality of $\mathfrak{C}$ is clearly $q^{n-1}(q-1)$. And each element of $\mathfrak{C}$ occurs as C_g for some $g \in G$ (think of the companion matrix). Since C_g is an invariant of the conjugacy class of g, the association $g \mapsto C_g$ gives rise to a map onto $\mathfrak{C}$ from the set of conjugacy classes of G.

We will now argue that when restricted to semisimple conjugacy classes the above association is a bijection. If $g = su$ be the Jordan decomposition, then $C_g = C_s$, so the restricted association is a surjection. On the other hand, if two semisimple matrices have the same characteristic polynomial, then they are conjugate (since they are conjugate on passing to $\overline{\mathbb{F}}_q$), and we are done.