TUTORIAL SHEET 12
SUNDRY ALGEBRAIC PRELIMINARIES

Let A be a ring (with identity as always, but with no further condition except any that is explicitly specified).

Essential surjections

Consider the following property of an A-linear map of A-modules:

(*) the image of every proper submodule is proper

Observe that the surjectivity of a map amounts to the pre-image of every proper submodule being proper. Surjections satisfying (*) are called *essential surjections*. A surjection $f : M \to N$ is essential if its kernel K has the property that $M' + K = M$ for a submodule M' of M implies $M' = M$. An essential surjection that splits is an isomorphism.

Let f and g be A-linear maps and fg their composition (g followed by f). Then

- If fg is surjective, so is f.
- If fg satisfies (*), so does g.
- If f and g satisfy (*), so does fg.
- If f and g are surjective, so if fg.
- If fg satisfies (*) and g is surjective, then f satisfies (*).
- If fg is surjective and f satisfies (*), then g is surjective.

Projective covers

A *projective cover* of an A-module M is a projective A-module P along with an A-linear essential surjection $P \twoheadrightarrow M$.

- Any two projective covers of a given module are isomorphic.
- If P is finitely generated projective module, then $P \twoheadrightarrow P/\mathfrak{Rad}\,P$ is a projective cover of $P/\mathfrak{Rad}\,P$.
- A finite direct sum of essential surjections is an essential surjection. (Hint: Express the direct sum of two essential surjections as a composition of two essential surjections.)

Essential injections

Consider the following property of an A-linear map of A-modules:

(†) the pre-image of every non-zero submodule is non-zero

Observe that the injectivity of a map amounts to the image of every non-zero submodule being non-zero. Injections satisfying (†) are called *essential injections*. An essential injection that splits is an isomorphism.

Injective hulls

An *injective hull* of an A-module M is an injective A-module I along with an A-linear essential injection $M \hookrightarrow I$.

- Any two injective hulls of a given module are isomorphic.