TUTORIAL SHEET 11
RADICALS

Let A be a ring and M an A-module. Let $\mathfrak{Rad}\, M$ denote the radical of M, and A_M the ring of homotheties of M (as an A-module).

(1) Let Z be the centre of A. Show that $Z \cap \mathfrak{Rad}\, A$ is contained in $\mathfrak{Rad}\, Z$.
(2) Show that $\mathfrak{Rad}\, A$ does not contain any idempotent other than 0, 1.
(3) If $\mathfrak{Rad}\, M = 0$, then $\mathfrak{Rad}\, A_M = 0$.
(4) If $A/\mathfrak{Rad}\, A$ is semisimple, then $\mathfrak{Rad}\, M = (\mathfrak{Rad}\, A)M$.

Let p be a prime, k a field of characteristic p, and G a finite group. Denote by kG the group ring of G over k. As always, $\mathfrak{Rad}$ is used to denote radical.

(1) Let $T_n(k)$ denote the ring of $n \times n$ upper triangular matrices with entries in a field k. Compute its radical.
(2) Suppose p divides the order of G. Let σ denote the element $\sum_{g \in G} g$ in kG. Then $k\sigma$ is a two sided nilpotent ideal of kG.
(3) For G a p-group, $\Delta(G) = \mathfrak{Rad}\, G$, where $\Delta(G)$ is the kernel of the map $kG \to k$ defining the trivial representation.
(4) If N is a normal subgroup of G, then $\mathfrak{Rad}\,(kN) = kN \cap \mathfrak{Rad}\,(kG)$.
(5) If N is a normal subgroup of G and T a simple kN-module, then there exists a simple kG-module S such that T is a direct summand of $S|_N$.