TUTORIAL SHEET 1

(1) Show that the centre of $\mathrm{GL}_k(V)$ is k^*.

(2) Let X be a G-set. The orbit Gx of a point x in X is isomorphic as a G-set to G/G_x, where $G_x := \{g \in G \mid gx = x\}$ is the *isotropy subgroup* at x. In particular, the cardinality of the orbit of a finite group divides the order of the group.

(3) Every irreducible linear representation of a finite group over a field is finite dimensional over the field.

(4) Giving a linear representation of a group G on a k-vector space is the same as giving a k-algebra homomorphism from the group ring kG to $\mathrm{End}_k V$.

(5) We can "twist" a representation by an automorphism (of the group) to get a new representation. This is a special case of "inflation". Such properties as irreducibility and semisimplicity are preserved under twisting. Twisting by an inner automorphism gives only an equivalent representation.

(6) (Clifford) Restriction to a normal subgroup of a semisimple representation is semisimple.

(7) Give an example of a group representation that is simple but when restricted to a subgroup is not semisimple. (Hint: Don't try a representation of a finite group over a field of characteristic zero.)

(8) (Averaging) For a linear representation V of a finite group G, write $V^G := \{v \in G \mid gv = v \text{ for all } G\}$. The map $v \mapsto \sum_{g\in G} gv$ is a G-map from $V \to V^G$. If $|G|$ is a unit in the field k, then $v \mapsto (1/|G|)\sum_{g\in G} gv$ is a G-projection onto V^G.

(9) New representations from old: If V and W are G-modules, then so is $\mathrm{Hom}(V, W)$: ${}^g\phi := g\phi g^{-1}$. If V is finite dimensional, then $\mathrm{Hom}(V, W) \simeq V^* \otimes W$ naturally; in fact, this is a natural G-isomorphism. Observe that $\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G$ where $\mathrm{Hom}_G(V, W)$ is the space of G-linear maps between V and W.