PREREQUISITE FOR MODULAR REPRESENTATION THEORY

ANUPAM SINGH

1. Modules of Interest

We denote A or R for a ring (not necessarily commutative) with 1. All modules are left module and typically denoted by symbols V and M. We always consider finite group typically denoted as G.

Example 1.1 (Group Algebra)**.** In this workshop we are going to worry about the following ring: Let k be a field and G be a finite group. We denote kG for the group algebra of G over the field k. This is a finite dimensional vector space over k. We will study structure theory of modules over this ring.

Exercise 1.2. Identify the group algebra in the case G is a cyclic group, S_3 or Q_8 over fields $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.

Example 1.3. Let D be a division ring (skew field). Then $M_n(D)$ is a ring.

A nonzero A-module M is called **simple (or irreducible)** if it contains no proper nonzero submodule. An A-module M is called **cyclic** with generator m if $M = Rm$ for some $m \in M$.

Exercise 1.4. Find out all simple modules over rings F, $\mathbb{Z}$ and $F[X]$ where F is a field.

Exercise 1.5. Let V be a vector space over a field F. Prove that V is a simple $\mathrm{End}_F(V)$-module.

Exercise 1.6 (Schur's Lemma)**.** Let M be a simple module and let $\phi \colon M \to M$ be a homomorphism. Prove that either $\phi = 0$ or ϕ is invertible.

Proposition 1.7. *For an A-module M, the following are equivalent.*

1. *M is the sum of a family of simple submodules.*
2. *M is the direct sum of a family of simple submodules.*
3. *Every submodule N of M is a direct summand.*

An A module M satisfying one of the above equivalent conditions is called **semisimple**.

Lemma 1.8. *Let $M = \sum_{i \in I} E_i$, E_i simple. Then there exists a subset $J \subset I$ such that $E = \oplus_{i \in J} E_i$.*

Lemma 1.9. *Let $M \neq 0$ be a module with property that every submodule is direct summand. Then every submodule of M contains a simple submodule.*

Example 1.10. Every module over a field F or a division ring D is semisimple.

Exercise 1.11. Find out when a cyclic module over rings $\mathbb{Z}$ or $F[X]$ is semisimple.

Example 1.12. kG over itself is semisimple if and only if $|G| \neq 0$ in k.

Exercise 1.13. Every submodule and quotient module of a semisimple module is again semisimple.

An A-module V is called **decomposable** if $V = V_1 \oplus V_2$ where V_i's are nonzero submodules. A nonzero module V is called **indecomposable** if it is not decomposable.

Exercise 1.14.
(1) All simple modules are indecomposable.
(2) Over $A = \mathbb{Z}$ the modules $V = \mathbb{Z}/p^r\mathbb{Z}$ where p is a prime, are indecomposable modules. However it is not simple for $r \geq 2$. Classify when $\mathbb{Z}/n\mathbb{Z}$ is an indecomposable $\mathbb{Z}$-module.
(3) Consider the group algebra A of cyclic group of order p over $k = \mathbb{Z}/p\mathbb{Z}$. Then A over itself is indecomposable but not simple. It is true in general that the group ring of a p-groups over a field of characteristic p is indecomposable.

Exercise 1.15. Let V be a finite dimensional vector space over a field k. Let $T \in GL(V)$. We make V a $k[X]$-module where $X.v := T(v)$. We denote this module by V_T.
(1) Let $T, S \in GL(V)$. Prove that $V_T \cong V_S$ as $k[X]$-module if and only if T and S are conjugate in $GL(V)$.
(2) Determine when V is a simple, semisimple, indecomposable and cyclic module. The answer should depend on T only.

2. Chain Conditions on Modules

Let V be an A-module. Then V is called **Artinian** if it satisfies the descending chain condition (DCC), i.e., every descending chain of submodules ($V \supset V_1 \supset V_2 \supset \cdots$) has finite length (i.e. after a fixed N all V_i are equal).

The module V is called **Noetherian** if it satisfies ascending chain condition (ACC), i.e., if every ascending chain of submodules ($0 \subset V_1 \subset V_2 \subset \cdots$) has finite length (i.e. after a fixed N all V_i are equal).

Example 2.1.
(1) Every module with finitely many elements over a ring satisfies both ACC and DCC, for example $\mathbb{Z}/n\mathbb{Z}$.
(2) $\mathbb{Z}$ over itself doesn't satisfy DCC but it satisfies ACC.
(3) Consider $V = (\mathbb{Q}/\mathbb{Z})_p$ (all elements of which order is a power of p) as a $\mathbb{Z}$ module. Then V has a unique submodule V_n of order p^n and $V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots$ does not satisfy ACC. However it satisfies DCC.
(4) $k[X]$ over itself satisfies ACC but not DCC.
(5) The ring $k[x_1, x_2, \ldots]$ in infinite variables doesn't satisfy either chain conditions.

Exercise 2.2. What about modules over kG?

Exercise 2.3. Determine whether simple (and semisimple) modules over a ring are Artinian or Noetherian?

The following lemmas give equivalent ways of defining Artinian and Noetherian of which proof is left as an exercise.

Lemma 2.4 (Artinian Module)**.** *Let V be an A module. Then the following are equivalent:*
(1) *V is Artinian.*
(2) *Every nonempty collection of submodules has a minimal element.*

Lemma 2.5 (Noetherian Module)**.** *Let V be an A module. Then the following are equivalent:*
(1) *V is Noetherian.*
(2) *Every nonempty collection of submodules has a maximal element.*
(3) *Every submodule of V is finitely generated.*

Corollary 2.6. *Let $0 \to V' \to V \to V'' \to 0$ be an exact sequence of A-modules. Then*

(1) *V is Artinian if and only if V' and V'' are Artinian.*
(2) *V is Noetherian if and only if V' and V'' are Noetherian.*

A ring A is called **Artinian or Noetherian** if $V = A$ is Artinian or Noetherian (left) A-module respectively.

Example 2.7. (1) A field k is Artinian as well as Noetherian.
(2) The ring $\mathbb{Z}/n\mathbb{Z}$ is also both Artinian and Noetherian.
(3) The ring $\mathbb{Z}$ is Noetherian but not Artinian.
(4) The algebra kG is Artinian and Noetherian both.

Proposition 2.8. *Let A be a Noetherian (resp. Artinian) ring. Then*

(1) *Finite direct sum of Noetherian modules is Noetherian.*
(2) *Every finitely generated module V over A is Noetherian (resp. Artinian).*
(3) *For an ideal I of A the ring A/I is Noetherian (resp. Artinian).*

Theorem 2.9. *An Artinian ring is Noetherian.*

3. Composition Series

A **chain of submodules** of a module V is a finite sequence $(V_i)_{0\leq i\leq n}$ of submodules of V such that

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V.$$

The modules V_{i+1}/V_i are the factor of the chain and n is called the length. A chain is called without repetition if 0 is not a factor. A chain (U_j) is a refinement of (W_i) if $0 = U_0 \subseteq \cdots \subseteq U_m$ is obtained from $0 = W_0 \subseteq \cdots \subseteq W_n$ by insertion of possibly some extra modules, i.e., it at least keeps all of the W_i's. A chain of submodules (V_i) is called a **composition series** if it is a chain without repetition and every proper refinement has repetition. Two chains are called **equivalent** if the factors of the chains (repetition allowed) are isomorphic upto some reordering.

Theorem 3.1. *Let V be an A module.*

(1) *(Schreier) Any two chains of submodules of V has equivalent refinements.*
(2) *(Jordan-Hölder) Any two composition series are equivalent.*

Proof. Step 1 : Consider two chains $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_m = V$ and $0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = V$ and define,

$$W_{i,j} = W_i + (W_{i+1} \cap U_j), \quad U_{j,i} = U_j + (U_{j+1} \cap W_i).$$

Step 2: $(W_{i,j})$ and $(U_{j,i})$ are refinements of (W_i) and (U_j) respectively. Also note that $W_{i,j+1} = W_{i,j} + (W_{i+1} \cap U_{j+1})$ and a similar relation for $U_{j,i+1}$. Now use isomorphism theorem to get the following:

$$\frac{W_{i,j+1}}{W_{i,j}} = \frac{W_{i+1} \cap U_{j+1}}{(W_{i+1} \cap U_{j+1}) \cap [W_i + (W_{i+1} \cap U_j)]}$$
$$\frac{U_{j,i+1}}{W_{j,i}} = \frac{W_{i+1} \cap U_{j+1}}{(W_{i+1} \cap U_{j+1}) \cap [U_j + (U_{j+1} \cap W_i)]}.$$

Step 3 : Prove that the denominators of both in the above equations are same. $\square$

Proposition 3.2. *Let V be an A module. Then V has a composition series if and only if V is Artinian as well as Noetherian.*

Proof. If V has a composition series no chain can have a larger length hence V is Artinian as well as Noetherian.

Conversely, suppose V is both Artinian and Noetherian. Then V has a minimal nonzero submodule (since Artinian), say V_1. Now V/V_1 is again Artinian and hence has a minimal submodule V_2/V_1. This way we form a chain:

$$0 \subset V_1 \subset V_2 \subset \cdots$$

which has to end in finitely many steps since V is also Noetherian. □

Let V be a module which has a composition series. The **length of module** is the length of composition series.

Exercise 3.3. Let V be a semisimple A module. Then prove that the following are equivalent:

(1) V is a direct sum of finitely many simple modules.
(2) V is Artinian.
(3) V is Noetherian.
(4) V has a composition series.

4. Radical and Socle

In this section we assume A is an Artinian ring and V is a finitely generated A-module. The **radical** of A, denoted as $rad(A)$, consists of elements of A which annihilate each simple A-module equivalently each semisimple A-module.

Exercise 4.1. $rad(A)$ is an ideal (note that A need not be commutative).

Proposition 4.2. *The radical of A is equal to each of the following:*

(1) *the smallest submodule of A whose corresponding quotient is semisimple;*
(2) *the intersection of all the maximal submodules of A;*
(3) *the largest nilpotent ideal of A.*

The **radical** of an A-module V denoted as $rad(V)$ is the intersection of all maximal submodules of V.

Proposition 4.3. *The following are equal to $rad(V)$:*

(1) $rad(A)V$;
(2) *the smallest submodule of V with semisimple quotient.*

Exercise 4.4. (1) If V is semisimple, $rad(V) = 0$.
(2) If V is Artinian and $rad(V) = 0$ then V is semisimple.

For an A-module V we define $rad^2(V) := rad(rad(V)) = rad(A)(rad(A)V) = (radA)^2V$. Inductively we define $rad^n(V) = rad(rad^{n-1}V)$. The sequence of modules:

$$V = rad^0(V) \supseteq rad^1(V) \supseteq rad^2(V) \supseteq \cdots$$

is called the **radical series** of V.

The **socle** of an A-module V, denoted as $soc(V)$, is the sum of all its simple (irreducible) submodules. If V has no simple submodule, its socle is (0).

Proposition 4.5. *The following are equal:*

(1) $soc(V)$;
(2) $\{v \in V \mid rad(A)v = 0\}$;
(3) *the largest semisimple submodule of V.*

Now we look at $V/soc(V)$ and denote its socale by $soc^2(V)/soc(V)$. Inductively we define $soc^n(V)$ and get a series:

$$0 = soc^0(V) \subseteq soc^1(V) = soc(V) \subseteq soc^2(V) \subseteq \cdots$$

called **socle series**.

5. Krull-Schmidt Unique Decomposition Theorem

Suppose V is a direct sum of indecomposables, i.e., $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$. Then it gives rise to certain elements $\pi_i \in \mathrm{End}_A(V)$ called projections with property $1 = \pi_1 + \pi_2 + \cdots + \pi_n$, $\pi_i^2 = \pi_i, \pi_i\pi_j = \pi_j\pi_i$ and $\pi_i(V) = V_i \subset V$. The following proposition guarantees existence of such elements.

Proposition 5.1. *If V is either Artinian or Noetherian then V is a finite direct sum of indecomposable modules.*

Proof. Let V be Noetherian. Consider the collection $\mathcal{F}$ consisting of submodules W of V such that V/W is not a finite direct sum of indecomposables. If $\mathcal{F}$ is nonempty then by Noetherian property we have maximal element in $\mathcal{F}$, say U. We have, V/U is not a finite direct sum of indecomposables and hence not indecomposable. Let $V/U = V_1/U \oplus V_2/U$ with nontrivial components. Then by maximality of U, we get $V/V_1 \cong V_2/U$ and $V/V_2 \cong V_1/U$ are finite direct sum of indecomposables and hence so is $V/U \cong V/V_1 \oplus V/V_2$, a contradiction. Thus the collection $\mathcal{F}$ is empty and $V \cong V/\{0\}$ is a finite direct sum of indecomposables.

Now let us assume that V is Artinian. Let $\mathcal{C}$ be the collection of all nonzero submodules of V which is not a finite direct sum of indecomposable submodules. If $\mathcal{C}$ is nonempty it will have a minimal element (as V is Artinian), say W. Clearly $W \neq 0$ and in not indecomposable hence $W = W_1 \oplus W_2$ where both W_1 and W_2 are nonzero. As W is minimal W_1 and W_2 both do not belong to $\mathcal{C}$ and are finite direct sum of indecomposable and hence so is W, a contradiction. This implies $\mathcal{C}$ is empty and hence V is a finite direct sum of indecomposables. □

Let A be a finite dimensional algebra over a field k. Then A is said to be **local** if every element of A is either nilpotent or invertible.

Exercise 5.2. Prove that A is local if and only if $A/rad(A) \cong k$. Also the set of all nilpotent elements form a two-sided ideal.

Proof. Let I be the set of all nilpotent elements. Let $u \in I$ and $a \in A$. We show that ua and au belong to I. Let n be the smallest integer such that $u^n = 0$. On contrary let us assume ua is not nilpotent hence they are invertible (as they are in the local ring A). Hence there exists $b \in A$ such that $uab = 1$. But then $u^{n-1} = u^{n-1}.1 = u^{n-1}.uab = 0$ which contradicts that n is smallest. Hence ua and au are nilpotent and belong to I.

Now let $u, v \in I$. Then $u + v \in A$ and are either nilpotent or invertible. Suppose they were invertible then there exists $a \in A$ such that $(u + v)a = 1$, i.e., $ua = 1 - va$. But ua is nilpotent and $1 - va$ is invertible (as it is 1 minus some nilpotent) which is a contradiction as both are equal. □

Now we are going to give different situations where $\mathrm{End}_A(V)$ is local. We will use this to prove the Krull-Schmidt theorem.

Theorem 5.3.
(1) *Let A be a finite dimensional k algebra. The A-module V is indecomposable if and only if $\mathrm{End}_A(V)$ is local.*
(2) *Let M be a module of finite length, i.e, Artinian and Noetherian both and suppose it is indecomposable. Then $\mathrm{End}_A(V)$ is local.*

(3) *Let A be a Noetherian ring and $A/rad(A)$ is Artinian and also assume that A is complete on Modules. Let V be a finitely generated indecomposable A-module. Then $\mathrm{End}_A(V)$ is local.*

Proof. If V is decomposable we can write $V = V_1 \oplus V_2$ and take the projection π. Then we have $\pi^2 = \pi$ where π is neither nilpotent nor unit.

Conversely, let $\rho \in \mathrm{End}_A(V)$ is neither nilpotent nor invertible. Let $\chi(x)$ be the characteristic polynomial of ρ. Then $\chi(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ has $a_0 = 0$ and is not of the form x^n, i.e., $\chi(x) = x^r f(x)$ where $f(0) \neq 0$. This nontrivial factorization of the characteristic polynomial provides a decomposition of V which is ρ invariant.

For the proof of part 2 see Curtis-Reiner or the exercise 5.6 and for the proof of part 3 see Dornhoff part B. □

Theorem 5.4 (Krull-Schmidt Theorem)**.** *Let V be an A-module where A is a finite dimensional k algebra (or satisfying one of the conditions in the above theorem so that $\mathrm{End}_A(V)$ is local). Suppose $V = U_1 \oplus \cdots \oplus U_r$ as well as $M = V_1 \oplus \cdots \oplus V_s$ are two decompositions into direct sum of indecomposable modules then $r = s$ and after suitable renumbering, $U_i \cong V_i$ for all i.*

Proof. We prove it by induction. Let π_i be the projections corresponding to the first decomposition and ρ_j for the second. We also have $I = \sum \pi_i = \sum \rho_j$ hence $\pi_1 = I.\pi_1 = \pi_1\rho_1 + \cdots + \pi_1\rho_s$. The restriction of $\pi_1\rho_j$ to U_1 is either nilpotent or invertible. If all of the $\pi_1\rho_j$ are nilpotent they belong to the ideal in $\mathrm{End}_A(U_1)$ consisting of all nilpotent elements and hence so is their sum, i.e., π_1 restricted to U_1. But that is identity map on U_1 which is not nilpotent. Hence one of the $\pi_1\rho_j$ is invertible. After renumbering of V_j's we may assume $\pi_1\rho_1$ is invertible restricted to U_1.

We look at the restricted maps as follows: $U_1 \xrightarrow{\rho_1} V_1 \xrightarrow{\pi_1} U_1$. Let $W = Im(\rho_1)$ and $K = ker(\pi_1)$. Since $\pi_1\rho_1$ is an isomorphism W is isomorphic to U_1. We claim that $K = 0$. For this first we show that $V_1 = W \oplus K$. Let $v \in V_1$ then there exists $w \in W$ such that $\pi_1(v) = \pi_1(w)$ and hence $v - w \in K$. Thus $v = w + (v - w) \in W + K$. Also if $v \in W \cap K$ then $\pi_1(v) = \pi_1(w)$ as π_1 is an isomorphism but since $v \in K$ as well we get $v = 0$. This proves $V_1 = W \oplus K$ and since π_1 is an isomorphism from W to U_1 we get $K = 0$.

Now we claim that $U_1 \cap (V_2 \oplus \cdots \oplus V_s) = 0$. This follows from dimension count. Hence we have proved that V_1 and U_1 are isomorphism and then by looking at M/U_1 we can use induction. □

Remark 5.5.
(1) There are possibly more indecomposable modules than simple modules over a ring.
(2) In the case $A = kG$ if $|G| \neq 0$ in k all modules are semisimple, i.e., direct sum of simples. However in the other case it is not so but still we have Krull-Schmidt decomposition.

Exercise 5.6. The following set of exercises provide proof for the part 2 in the theorem 5.3.

(1) A surjective endomorphism of a Noetherian module is bijective. (Hint: Consider the chain $0 \subseteq ker(u) \subseteq ker(u^2) \subseteq \cdots$ which stabilises at stage n. Then prove that $ker(u^n) \cap Im(u^n) = 0$. Prove that u^n is injective and hence so is u).
(2) An injective endomorphism of an Artinian module M is bijective. (Hint: Consider the chain $M \supseteq Im(u) \supseteq Im(u^2) \supseteq \cdots$ which stabilises at stage n. Prove that $M = ker(u^n) + Im(u^n)$).
(3) (**Fitting Lemma :**) Let M be a module which is Artinian and Noetherian both. Let $u \in \mathrm{End}_A(M)$. Prove that there exists n such that $M = ker(u^n) \oplus Im(u^n)$.
(4) Let V be a module of finite length. Then V is indecomposable if and only if $\mathrm{End}_A(V)$ is local.

6. Projective and Injective Modules

A module V isomorphic to $A \oplus \cdots \oplus A$ is called a free module, i.e., there exists a subset $X \subset V$ such that every $v \in V$ has unique expression of the form $v = r_1 v_1 + \cdots + r_n v_n$ where $r_i \in A$ and $v_i \in X$.

Proposition 6.1 (Free Modules)**.** *An A-module V is free over a set X if and only if it satisfies the following universal property: For any A-module M and set map $\phi\colon X \to M$ there exists a unique A-module homomorphism $\Phi\colon V \to M$ such that $\Phi(v) = \phi(v)$ for all $v \in X$.*

Exercise 6.2 (Split exact sequence)**.** The following are equivalent for a short exact sequence $0 \to L \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0$:

(1) there exists a homomorphism $\alpha\colon N \to M$ such that $\psi\alpha = id$.
(2) there exists a homomorphism $\beta\colon M \to L$ such that $\beta\phi = id$.
(3) there exists $N' \subset M$ such that $M = \phi(L) \oplus N'$.

Exercise 6.3. (1) The functor $\mathrm{Hom}_A(P, -)$ is left exact, i.e., for any short exact sequence of A modules

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\phi} N \to 0$$

the sequence

$$0 \to \mathrm{Hom}_A(P, L) \xrightarrow{\psi'} \mathrm{Hom}_A(P, M) \xrightarrow{\phi'} \mathrm{Hom}_A(P, N)$$

is also exact where ψ' and ϕ' are natural composition maps.

(2) The functor $\mathrm{Hom}_A(-, Q)$ is left exact, i.e., for any short exact sequence of A modules

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\phi} N \to 0$$

the sequence

$$0 \to \mathrm{Hom}_A(N, Q) \xrightarrow{\phi'} \mathrm{Hom}_A(M, Q) \xrightarrow{\psi'} \mathrm{Hom}_A(L, Q)$$

is also exact.

Proposition 6.4 (Projective Modules)**.** *Let P be an A-module. Then the following are equivalent:*

(1) *The functor $\mathrm{Hom}_A(P, -)$ is exact, i.e., for any short exact sequence of A modules*

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\phi} N \to 0$$

the sequence

$$0 \to \mathrm{Hom}_A(P, L) \xrightarrow{\psi'} \mathrm{Hom}_A(P, M) \xrightarrow{\phi'} \mathrm{Hom}_A(P, N) \to 0$$

is also exact where ψ' and ϕ' are natural composition maps.

(2) *For any M, N if sequence of A modules $M \xrightarrow{\phi} N \to 0$ is exact, then any homomorphism f from P to N lifts to an A module homomorphism F from P to M.*

$$\begin{tikzcd} & P \arrow[dl, dashed, "F"'] \arrow[d, "f"] & \\ M \arrow[r, "\phi"] & N \arrow[r] & 0 \end{tikzcd}$$

(3) *Every short exact sequence $0 \to L \to M \to P \to 0$ splits, i.e., if P is a quotient of the A module M then P is isomorphic to a direct summand of M.*

(4) *P is a direct summand of a free A-module.*

Proof. $(1 \Leftrightarrow 2)$: If we have ψ injective we always get ψ' injective. And the surjectivity of ϕ' is equivalent to the statement 2.

$(2 \Rightarrow 3)$: Let us consider the exact sequence $M \to P \to 0$ and the identity map $I\colon P \to P$. From 2 it lifts to a map from P to M which provides the splitting.

$(3 \Rightarrow 4)$: Every module is a quotient of a free module. Hence have $P \cong F/K$ where F is a free module. This gives rise to the exact sequence $0 \to K \to F \to P \to 0$ which splits by hypothesis 3. Hence $P \oplus K \cong F$.

$(4 \Rightarrow 2)$: Suppose $P \oplus K \cong F$ where F is free on a set X. Then we make the following diagram:

$$
\begin{tikzcd}
 & F \cong P \oplus K \arrow[d, "\pi"] \arrow[ddl, dashed, "F'"'] & \\
 & P \arrow[d, "f"] & \\
M \arrow[r, "\phi"] & N \arrow[r] & 0
\end{tikzcd}
$$

The map $f \circ \pi$ is defined on the free module F hence equivalent to being defined on X. We first define F' on X by looking at the inverse under ϕ of $f \circ \pi$ of elements of X and uniquely extend it to define F'. The restriction of F' on P does the job. $\square$

Any A-module P satisfying the above equivalent conditions is called **projective**.

Example 6.5. Let k be a field. Then any module V (which is a vector space) is a free module hence projective. In fact, every free module over a ring A is projective.

Example 6.6. Non zero finite Abelian groups are not projective $\mathbb{Z}$ modules. A finitely generated $\mathbb{Z}$ module is projective if and only if it is free.

Example 6.7. The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is not projective. As the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \to 0$ does not split.

Example 6.8. In a Dedekind domain an ideal which is not principal is an example of a projective module which is not free.

Exercise 6.9. Prove that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

Exercise 6.10. Direct summands and direct sums of projectives are projectives.

Exercise 6.11. Every module is a quotient of a free module and hence also of a projective module.

Proposition 6.12 (Injective Modules)**.** *Let Q be an A-module. Then the following are equivalent:*

(1) *The functor $\mathrm{Hom}_A(-, Q)$ is exact, i.e., for any short exact sequence of A modules*

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\phi} N \to 0$$

the sequence

$$0 \to \mathrm{Hom}_A(N, Q) \xrightarrow{\phi'} \mathrm{Hom}_A(M, Q) \xrightarrow{\psi'} \mathrm{Hom}_A(L, Q) \to 0$$

is also exact.

(2) *For any L, M if sequence of A modules $0 \to L \xrightarrow{\phi} M$ is exact, then any homomorphism f from L to Q lifts to an A module homomorphism F from M to Q.*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\phi"] \arrow[d, "f"'] & M \arrow[dl, dashed, "F"] \\ & Q & \end{tikzcd}$$

(3) *Every short exact sequence $0 \to Q \to M \to N \to 0$ splits, i.e., if Q is a submodule of the A-module M then Q is a direct summand of M.*

An A-module Q satisfying the above equivalent conditions is called **injective**.

Example 6.13. $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module since the exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ is not split.

A $\mathbb{Z}$-module R is said to be **divisible** if $R = nR$ for all nonzero integers n.

Proposition 6.14. *Let Q be an A module.*

(1) **Baer's criterion :** *The module Q is injective if and only if for every left ideal I of A and A-module homomorphism $g\colon I \to Q$ can be extended to an A-module homomorphism $G\colon A \to Q$.*
(2) *If A is a PID then Q is injective if and only if $rQ = Q$ for every nonzero $r \in A$. In particular, a $\mathbb{Z}$-module is injective if and only if it is divisible. When A is a PID, quotient modules of injective A-modules are again injective.*

Example 6.15. $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules.

Example 6.16. Arbitrary direct products of injectives are injectives. Arbitrary direct sums of injectives are injective over Noetherian rings.

Example 6.17. No non-zero finitely generated $\mathbb{Z}$-module is injective.

Example 6.18. Let k be a field. Then every module over ring $M_n(k)$ is injective as well as projective.

Example 6.19. Let k be a field and G a finite group of order n. Suppose $n \neq 0$ in k. Then every module over the group ring kG is injective as well as projective.

Example 6.20. A ring A is called **semisimple** if A is a semisimple module over itself. Every module over a semisimple ring is injective as well as projective.

7. Relatively Projective Modules

We give another characterization of free module in a special case as follows:

Proposition 7.1. *Let A be a finite dimensional k-algebra. An A-module U is free if and only if U has a subspace X such that any linear transformation from X to any A-module V extends uniquely to a module homomorphism of U to V.*

Let B be a subring of A. We say that an A-module U is **relatively B-free** if there exists a B-submodule X of U such that any B-homomorphism of X to any A-module V extends uniquely to an A-homomorphism of U to V. The typical situation for this is A-a finite dimensional k-algebra and B a subalgebra. Observe that if $B = k$, the trivial subalgebra, relatively free is same as free.

Exercise 7.2. Let H be a subgroup of G. Take $A = kG$ and $B = kH$. Prove that there exists a relatively H-free kG-module. Let V be an H-module. Then $Ind_H^G V = kG \otimes_{kH} V$ is relatively free module.

Let B be a subring of A. We consider the situation when B is Noetherian and A is a finitely generated B-module. A finitely generated A-module P is called **relatively B-projective** if for any finitely generated A-modules M, N the exact sequence

$$0 \to M \xrightarrow{\phi} N \xrightarrow{\psi} P \to 0$$

splits whenever the restricted sequence

$$0 \to M_B \xrightarrow{\phi} N_B \xrightarrow{\psi} P_B \to 0$$

(as B-modules) is a split exact sequence.

Exercise 7.3. Let A be finite dimensional algebra over a field k and B a subalgebra. Suppose A is a finitely generated B-free module. Prove that P is projective if and only if P_B is a projective B-module and P is relatively B projective.

Exercise 7.4. Let A be a finite dimensional k-algebra and V a finitely generated A-module. Then V is projective if and only if V is relatively k-projective.

Proposition 7.5 (Relatively Projective)**.** *Let A be a finite dimensional k algebra and B a subalgebra. Let U be an A-module. Then the following are equivalent:*

(1) $\mathrm{Hom}_A(U, -)$ *is exact provided* $\mathrm{Hom}_B(U, -)$ *is so, i.e., if the sequence of A modules*

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\phi} N \to 0$$

is exact then the sequence

$$0 \to \mathrm{Hom}_A(U, L) \xrightarrow{\psi'} \mathrm{Hom}_A(U, M) \xrightarrow{\phi'} \mathrm{Hom}_A(U, N) \to 0$$

is also exact provided it is so for B-module homomorphisms.

(2) *For any M, N; if the sequence of A modules $M \xrightarrow{\phi} N \to 0$ is exact, then any homomorphism $f : U \to N$ lifts to an A-module homomorphism $F : U \to M$ provided f lifts as B homomorphism.*

$$\begin{tikzcd} & U \arrow[dl, dashed, "F"'] \arrow[d, "f"] & \\ M \arrow[r, "\phi"] & N \arrow[r] & 0 \end{tikzcd}$$

(3) *U is relatively B-projective.*

(4) *U is a direct summand of a relatively B-free module.*

Proof. It is easy to verify that 1 and 2 are equivalent. □

8. Grothendieck Group

Let A be a ring and $\mathcal{F}$ be a category of finitely generated left A-modules. The **Grothendieck group** of $\mathcal{F}$, denoted as $K(\mathcal{F})$, is the Abelian group generated by $[E]$ for each $E \in \mathcal{F}$ and the relations are $[E'] = [E] + [E'']$ for each exact sequence $o \to E \to E' \to E'' \to 0$ in $\mathcal{F}$.

Proposition 8.1 (Universal Property)**.** *For any Abelian group H, and a map $\phi\colon \mathcal{F} \to H$ which is additive, i.e., $\phi(E') = \phi(E) + \phi(E'')$ for each exact sequence $o \to E \to E' \to E'' \to 0$, there exists a unique Abelian group homomorphism $f\colon K(\mathcal{F}) \to H$ sending $[E]$ to $\phi(E)$.*

Example 8.2. Let G be a finite group and k a field of characteristic 0. We consider the category of finite dimensional representations of G over k. Then the Grothendieck group in this case is the Abelian group $R_k(G) = \mathbb{Z}\chi_1 \oplus \cdots \mathbb{Z}\chi_r$ spanned by the irreducible characters. Moreover $R_k(G)$ is a ring.

Example 8.3.
(1) Let L be a field extension of k. Then we have a ring homomrphism $R_k(G) \to R_L(G)$ given by $V \mapsto V \otimes_k L$.
(2) Let H be a subgroup of G. Then we have a ring homomorphism $Ind\colon R_k(H) \to R_k(G)$ given by $V \mapsto Ind_H^G V$. The Frobenius reciprocity says that $Res\colon R_k(G) \to R_k(H)$ is adjoint of Ind.

Exercise 8.4. Let C be an algebraically closed field containing field k. Let G be a finite group. Then we have $R_k(G) \subset R_C(G)$. We define subset $\bar{R}_k(G) = \{\chi \in R_C(G) \mid Im(\chi) \subset k\}$ then we have

$$R_k(G) \subset \bar{R}_k(G) \subset R_C(G).$$

Then prove the following:

(1) Let V_i be distinct irreducible representations over k and χ_i be their characters. Then χ_i form a basis of $R_k(G)$ and are mutually orthogonal.
(2) A representation of G is realizable over k if and only if its character belong to $R_k(G)$.
(3) Show that every representation over $\mathbb{C}$ is realizable over $\bar{\mathbb{Q}}$ and hence over a number field.

Hint : Look at [Se] chapter 12 Proposition 32 and 33.

Example 8.5. Let k be a field. For the category of finite dimensional vector spaces the Grothendieck group is $\mathbb{Z}$ as any vector space is isomorphic to k^n for some $n \in \mathbb{N}$.

Exercise 8.6. Prove that the Grothendieck group for the category of finitely generated modules over $\mathbb{Z}$ is again $\mathbb{Z}$.

Exercise 8.7. Let G be a finite group and k be a field of characteristic 0. Consider the category of finitely generated projective G- modules. What is the Grothendieck group. (Direct sum of projectives is projective.)

Exercise 8.8 (Lagrange's Theorem)**.** Let P be a projective kG module and H a subgroup of G. Then P_H (as H-module) is a projective kH-module.

Exercise 8.9. Let k be an algebraically closed field of characteristic p. Prove that for the Abelian group $\mathbb{Z}/n\mathbb{Z}$ with $n = p^a r$ with $(r, p) = 1$ there are exactly r simple modules and n indecomposable modules.

Hint : Use Jordan canonical form theory.

Exercise 8.10. Let k be an algebraically closed field of characteristic p. Prove that for a p-group there is exactly 1 simple module.

References

[Al] Alperin, J. L., *"Local representation theory"*, Cambridge Studies in Advanced Mathematics, 11. Cambridge University Press, Cambridge, 1986.

[AB] Alperin, J. L.; Bell R. B., *"Groups and Representations"*, Graduate Texts in Mathematics, 162, Springer-Verlag, New York, 1995.

[CR1] Curtis C. W.; Reiner I., *"Methods of representation theory vol I, With applications to finite groups and orders"*, Pure and Applied Mathematics, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1981.

[CR2] Curtis C. W.; Reiner I., *"Methods of representation theory vol II, With applications to finite groups and orders"*, Pure and Applied Mathematics, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1987.

[CR3] Curtis C. W.; Reiner I., *"Representation theory of finite groups and associative algebras"*, Pure and Applied Mathematics, Vol. XI Interscience Publishers, a division of John Wiley & Sons, New York-London 1962.

[D1] Dornhoff L., *"Group representation theory. Part A: Ordinary representation theory"*, Pure and Applied Mathematics, 7. Marcel Dekker, Inc., New York, 1971.

[D2] Dornhoff, L., *"Group representation theory. Part B: Modular representation theory"*. Pure and Applied Mathematics, 7. Marcel Dekker, Inc., New York, 1972.

[Fe] Feit, Walter, *"The representation theory of finite groups"* North-Holland Mathematical Library, 25. North-Holland Publishing Co., Amsterdam-New York, 1982.

[FH] Fulton; Harris, *"Representation theory: A first course"* Graduate Texts in Mathematics, 129, Readings in Mathematics, Springer-Verlag, New York, 1991.

[Hu] Humphreys, James E., *"Modular representations of finite groups of Lie type"*, London Mathematical Society Lecture Note Series, 326. Cambridge University Press, Cambridge, 2006.

[Se] Serre J.P., *"Linear representations of finite groups"* Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42, Springer-Verlag, New York-Heidelberg, 1977..

[Si] Simon, B., *"Representations of finite and compact groups"* Graduate Studies in Mathematics, 10, American Mathematical Society, Providence, RI, 1996.

[M] Musili C. S., *"Representations of finite groups"* Texts and Readings in Mathematics, Hindustan Book Agency, Delhi, 1993.

[DF] Dummit D. S.; Foote R. M. *"Abstract algebra"*, Third edition, John Wiley & Sons, Inc., Hoboken, NJ, 2004.

[L] Lang, *"Algebra"*, Second edition, Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, 1984.

[JL] James, Gordon; Liebeck, Martin, *"Representations and characters of groups"*, Second edition, Cambridge University Press, New York, 2001.

IISER, central tower, Sai Trinity building, Pashan circle, Pune 411021 INDIA
E-mail address: `anupam@iiserpune.ac.in`