Outline of Group Theory: Week 4

Venketasubramanian C G

---

1. Introduction

This is an outline of four lectures delivered during the 4^{th} week (26 Dec 2022 to 31 Dec 2022) of the Annual Foundation School-I conducted by National Center for Mathematics at MEPCO Schlenk Engineering College, Sivakasi during December 2022. Each lecture was accompanied by a tutorial session. Most of the examples in these notes were used as problems in tutorial sessions. Apart from generalities, I have followed closely Chapter 8 from Serre's 'Finite Groups-an introduction', International Press, (2016). The aim of the lectures were to give a short and succinct introduction to the theory of complex representations of finite groups in four lectures.

Unless mentioned otherwise, we will assume that all our groups are finite and all vector spaces are over complex numbers and finite dimensional.

2. Definitions and Examples

Definition 2.1. — A representation of a group G is a pair (π, V) where V is a $\mathbb{C}$-vector space and

$$\pi : G \to Aut_{\mathbb{C}}(V)$$

is a group homomorphism i.e., $\pi(gh) = \pi(g) \circ \pi(h)\ \forall g, h \in G$.

Remark 2.2. — We shall assume in these notes that V has finite diemsnion.

If V is of dimension n and we fix a basis $\mathcal{B}$ of V, we get an isomorphism of $\mathbb{C}$-algebras:

$$\begin{aligned} \phi : End(V) &\xrightarrow{\cong} M_n(\mathbb{C}) \\ T &\mapsto [T]_{\mathcal{B}} \end{aligned}$$

Definition 2.3. — If (π, V) is a representation of group G, then degree of the representation is defined as follows:

$$\deg(\pi, V) := \dim(V)$$

It is customary to denote the degree by $\deg(\pi)$.

Example 2.4. — $G = (\mathbb{Z}/n\mathbb{Z}, +)$, then

$$\chi_k : (\mathbb{Z}/n\mathbb{Z}, +) \to (S^1, \cdot) \subset (\mathbb{C}^*, \cdot)$$

defined by

$$\chi_k(\bar{1}) = e^{\frac{2\pi i}{n}k}$$

for each k such that $0 \leqslant k \leqslant n-1$. Each of the χ_k gives a one-dimensional representation of $\mathbb{Z}/n\mathbb{Z}$.

Example 2.5. — Let $G = S_3$.
We get a trivial representation of S_3 given by $g \mapsto 1 \in \mathbb{C}^*$ for all $g \in S_3$.

Another one dimensional representation, *sign representation*

$$sgn : S_3 \to \mathbb{C}^*$$

is defined as

$$g \mapsto sgn(g)$$

where $\mathrm{sgn}(\sigma)$ denotes the sign of the permutation σ.
We define a degree 2 representation of S_3.
Write $S_3 = \{1, r, r^2, s, rs, r^2s\}$.
We map

$$r \mapsto \begin{bmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{bmatrix} \text{ and } s \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

where $n = 3$. This gives a representation $\pi : S_3 \to Aut(\mathbb{C}^2)$.

The next example is a recollection from Linear Algebra.

Example 2.6. — Suppose X is a finite set. Consider $(\mathcal{F}(X), +, \cdot)$ where $+$ is pointwise addition and $\cdot$ is scalar multiplication. Then,

1. $(\mathcal{F}(X), +, \cdot)$ is a $\mathbb{C}$ -vector space.
2. A basis of $\mathcal{F}(X)$ is given by $\{\delta_x : x \in X\}$ where δ_x is the function on X which takes the value 1 at x and 0 at points $y \neq x$.
3. $\dim(\mathcal{F}(X)) = |X|$.

Example 2.7. — Suppose G acts on X. Let $\mathcal{F}(X) := \{f : X \to \mathbb{C}\}$. Then G acts on $\mathcal{F}(X)$ as follows: $g \in G, x \in X$ $(g \cdot f)(x) = f(g^{-1} \cdot x)$. Lets denote the action of G on $\mathcal{F}(X)$ by Π. Then each $\Pi(g)$ is a linear map of the vector space $\mathcal{F}(X)$. Thus,

$$\Pi : G \to Aut(\mathcal{F}(X))$$

given by $[\Pi(g)f](x) = f(g^{-1} \cdot x)$ for $g \in G, f \in \mathcal{F}(X)$ and $x \in X$ is a representation of G on $\mathcal{F}(X)$.

We will apply the previous example to G acting on itself via left and right action.

Example 2.8. — Recall that G acts on G via:

(i) left action: $\lambda(g)x = g \cdot x$
(ii) right action: $\rho(g)x = x \cdot g^{-1}$

Applying the general construction in the previous example to λ and ρ, G acts on $\mathcal{F}(G)$ via λ and ρ. Let us write down the representations: $\lambda : G \to Aut(\mathcal{F}(G))$ and $\rho : G \to Aut(\mathcal{F}(G))$:

$$[\lambda(g)f](x) = f(g^{-1}x)$$

and

$$[\rho(g)f](x) = f(xg)$$

for $g \in G, f \in \mathcal{F}(G), x \in X$.

The representations $(\lambda, \mathcal{F}(G))$ and $(\rho, \mathcal{F}(G))$ are called respectively the left-regular and right regular representation of G.

Example 2.9. — Using λ and ρ, we can define an action of $G \times G$ on G: $(g, h) \cdot x = gxh^{-1}$ for $g, x, h \in G$. This induces a representation of $G \times G$ on $\mathcal{F}(G)$ denoted by $\lambda \times \rho : G \to Aut(\mathcal{F}(G))$ given by

$$[(\lambda \times \rho)(g, h)f](x) = f(g^{-1}xh)$$

for $g \in G, f \in \mathcal{F}(G), x \in X$.

We have an algebra structure on the vector space $\mathcal{F}(G)$.

Definition 2.10. — Let G be a finite group and let $f_1, f_2 \in \mathcal{F}(G)$. Define

$$f_1 * f_2 : G \to \mathbb{C}$$

by

$$(f_1 \star f_2)(x) = \frac{1}{|G|} \sum_{g \in G} f_1(g) f_2(g^{-1}x).$$

Lemma 2.11. — *1. The multiplication $*$ is associative.*
2. $(\mathcal{F}(G), +, \cdot, *)$ is a $\mathbb{C}$-algebra.
3. $(\mathcal{F}(G), +, *)$ is a ring with an identity $\widehat{\delta_g} := |G|\delta_g$.
4. For $g, h \in G$, $\widehat{\delta_g} * \widehat{\delta_h} = \widehat{\delta_{gh}}$.
5. The ring $(\mathcal{F}(G), +, *)$ is commutative if and only if the group G is abelian.

Exercise. Can you give an example of a one-dimensional representation of the group $GL_2(\mathbb{F}_p)$?(Hint: determinant). What are all the one dimensional representations of the group $GL_2(\mathbb{F}_p)$?

3. New representations from old and few basic results

We will see more examples of constructions of representations. First, we will apply the general construction of representation of the previous lecture to S_n.

Example 3.1. — Write $J_n = \{1, \ldots, n\}$. We can identify $\mathcal{F}(J_n)$ with $\mathbb{C}^n$. The action $\sigma \cdot f(j) = f(\sigma^{-1}(j))$ for $\sigma \in S_n, f \in \mathcal{F}(J_n)$ and $j \in J_n$ translates to an action of S_n on $\mathbb{C}^n$ via $\sigma \cdot (z_1, \ldots, z_n) = (z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(n)})$. Under this action, $\sigma \cdot e_j = e_{\sigma(j)}$. This is easy to see via functions if we identify the basis vector e_j with δ_j: $(\sigma \cdot \delta_j)(i) = 1$ if and only if $\sigma^{-1}(i) = j$ if and only if $i = \sigma(j)$. That is, $\sigma \cdot \delta_j = \delta_{\sigma(j)}$.

Example 3.2. — In this example, we construct the contragredient or dual of a representation. Suppose (π, V) is a representation of a group G. Define

$$V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) = \{l : V \to \mathbb{C} |\ l \text{ is linear}\} \subset \mathcal{F}(V)$$

Let G act on V by π, so we have $g \cdot v = \pi(g)v$. G acts on $\mathcal{F}(V)$ by the action

$$(g \cdot f)(v) = f(g^{-1} \cdot v) = f(\pi(g^{-1})v).$$

1. If $l \in V^*$ define $g \cdot l(v) = l(\pi(g^{-1}v))$
2. It is easy to check that $g \cdot l \in V^*$

Thus we get a natural action of G on V^* denoted by π^*:

$$(\pi^*(g)l)(v) = l(\pi(g^{-1}v))$$

(π^*, V^*) is called the *dual/contragradient representation* of (π, V). Note that V^* is a subspace of $\mathcal{F}(V)$ and has the following property: For any $\ell \in V^*$ and $g \in G$, $g \cdot l \in V^*$. We call such a subspace a G-invariant subspace of $\mathcal{F}(V)$. It allowed us to define a representation of G on the subspace V^*.

From the previous example, we are led to the notion of a subrepresentation.

Definition 3.3. — Let (π, V) be a representation of a group G. We say that a subspace W of V is G-invariant if for each $g \in G$, $\pi(g)(W) \subset W$.
This gives a representation of G on W.

$$\begin{aligned} \pi|_W : & G \to Aut(W) \\ & g \mapsto \pi(g) \end{aligned}$$

$(\pi|_W, W)$ is said to be a subrepresentation of (π, V).

Next, we construct representations using direct sums.

Example 3.4. — Suppose (π_1, V_1) and (π_2, V_2) are representations of G. Define

$$\pi_1 \oplus \pi_2 : G \to Aut(V_1 \oplus V_2)$$

defined by

$$(\pi_1 \oplus \pi_2)(g)(v_1, v_2) = (\pi_1(g)(v_1), \pi_2(g)(v_2))$$

It is easy to check that $(\pi_1 \oplus \pi_2, V_1 \oplus V_2)$ is a representation of G.

Example 3.5. — Let (π, V) and (τ, W) be representations of a group. We can define a representation on the vector space $\mathrm{Hom}_{\mathbb{C}}(V, W)$ consisting of all linear maps from V to W. Define for $\phi \in \mathrm{Hom}_{\mathbb{C}}(V, W)$, $g \in G$ and $v \in V$,

$$(g \cdot \phi)(v) = \tau(g)(\phi(\pi(g^{-1})(v)))$$

It is easy to check that this is indeed a representation of G. This representation will be denoted by $Hom(\pi, \tau)$.

Definition 3.6. — A representation (π, V) of G is said to be irreducible if $V \neq 0$ and V has no G-invariant subspaces other than V and 0.

Example 3.7. — Any degree one representation of a group is irreducible by definition.

Example 3.8. — The representation $(\pi, \mathbb{C}^2)$ of S_3 we defined in class is irreducible. What about the two dimensional representation of D_4 we defined in class?

Example 3.9. — Consider the left regular representation $(\lambda, \mathcal{F}(G))$ of G. Define W to be the subspace of $\mathcal{F}(G)$ consisting of all constant functions. Then, W is a G-invariant subspace of $\mathcal{F}(G)$. In fact, $\lambda(g)f = f$ for all $f \in W$ and $g \in G$. Thus (λ, W) is a one dimensional representation of G and hence irreducible. Note that W is spanned by the constant function 1. If G is a non-trivial group, then $(\lambda, \mathcal{F}(G))$ is not irreducible.

Example 3.10. — Consider the action of S_n on $\mathbb{C}_n$ via λ : from the previous example, the constant function 1 generates a one dimensional invariant subspace which can be identified with the subspace of $\mathbb{C}^n$ spanned by $e_1 + \cdots + e_n$.

Definition 3.11. — Suppose (π, V) is a finite dimensional representation of a group G. Define

$$V^G = \{v \in V : \pi(g)v = v \text{ for all } g \in G\}.$$

V^G is a G-invariant subspace known as the space of G-fixed vectors.

Example 3.12. — If (π, V) is a finite dimensional representation of a group G the space of G-fixed vectors is a subrepresentation of V.

Example 3.13. — Suppose (π, V) is a representation of a finite group G.

1. Let $v \in V$ be a non-zero vector. The smallest G-invariant subspace of V containing v is the subspace $W_v := \mathrm{Span}(\{\pi(g)v : g \in G\})$. Observe that the dimension of W_v is at most |G|.
2. If (π, V) is irreducible and $v \in V$ is non-zero, we have $W_v = V$.
3. If (π, V) is irreducible, we have $\deg(\pi) \leqslant |G|$. That is, the degree of any irreducible representation of a finite group cannot exceed the order of the group.

Our next aim is to prove a basic theorem in the subject, namely, any finite dimensional complex representation of a finite group can be written as a direct sum of its irreducible subrepresentations.

Remark 3.14. — Let V be an n-dimensional $\mathbb{C}$-vector space.We may identify V with $\mathbb{C}^n$ and transfer the standard inner-product $\langle z, w\rangle = \sum_{j=1}^{n} z_j \bar{w}_j$ on $\mathbb{C}^n$ to V.

Proposition 3.15. — *Suppose (π, V) is a representation of a finite group G. Then, there exists an inner product $\langle -, -\rangle$ on V such that $\langle \pi(g)v, \pi(g)w\rangle = \langle v, w\rangle$ $\forall g \in G, v, w \in V$. That is, $\langle -, -\rangle$ is a G-invariant inner product on V.*

Proof. — If $\langle , \rangle_0$ is any arbitrary inner product on G, define

$$\langle v, w\rangle := \frac{1}{|G|}\sum_{g\in G}\langle \pi(g)v, \pi(g)w\rangle$$

It is easy to verify that $\langle , \rangle$ is as claimed in the statement of the theorem. □

Proposition 3.16. — *Suppose (π, V) is a representation of a finite group G. Assume that W is a G-invariant subspace of V. Then, there exists a subspace U of V such that U is G-invariant and $V = W \oplus U$.*

Proof. — For any subspace W of V which is G-invariant consider its orthogonal complement $W^\perp$ with respect to the G-invariant inner product given by the previous the previous theorem. □

Theorem 3.17 **(Maaschke's theorem)**. — *Let (π, V) be a finite dimensionl repesentation of a finite group G. Then there exist irreducible subrepresentations, $W_1, \ldots, W_r$ of V such that*

$$V = W_1 \oplus \cdots \oplus W_r$$

Proof. — Apply induction on dimension of V. If $\dim(V) = 1$ there is nothing to prove. Assume that $\dim(V) > 1$. If (π, V) is irreducible then there is nothing to prove. Assume (π, V) is not irreducible to get a proper non-zero G-invariant subspace W. Apply previous result to get $V = W \oplus W^\perp$. Apply induction hypothesis to W and $W^\perp$. □

Example 3.18. — Assume G is finite. On $\mathcal{F}(G)$, define a form: $f_1, f_2 \in \mathcal{F}(G)$, define

$$\langle f_1, f_2\rangle = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)} \tag{3.1}$$

1. Show that $\langle , \rangle$ is a G-invariant inner product on $\mathcal{F}(G)$.
2. Let W denote the subspace of constant functions on G. Show that W is G-invariant.
3. What is a G-invariant complement of W in $\mathcal{F}(G)$?

4. Character of a group representation and its properties

Definition 4.1. — Let G be a finite group. Let (π, V) be a finite dimensional representation of the group G. Define $\chi_\pi : G \to \mathbb{C}$ by

$$\chi_\pi(g) = \mathrm{tr}(\pi(g)).$$

This invariant of the representation is called the character of the representation (π, V).

Remark 4.2. — Suppose $|G| = n$. Note that $g^n = e$ implies that $\pi(g)^n = I_V$. Therefore, $\pi(g)$ satisfies the polynomial $X^n - 1 \in \mathbb{C}[X]$, which has n distinct zeroes in $\mathbb{C}$. Hence there exists a basis $\mathcal{B}$ of V such that

$$[\pi(g)]_{\mathcal{B}} = \begin{bmatrix} \lambda_1(g) & & \\ & \ddots & \\ & & \lambda_n(g) \end{bmatrix}$$

Hence, $\chi_\pi(g) = \mathrm{tr}[\pi(g)]_{\mathcal{B}} = \lambda_1(g) + \cdots + \lambda_n(g)$. Since $\lambda_1(g), \ldots \lambda_n()$ are roots of $X^n - 1$, $\lambda_1(g), \ldots \lambda_n(g) \in S^1$.

Proposition 4.3. — *The character of a finite dimensional representation of a finite group has the following properties.*

1. *If $g, h \in G$, $\chi_\pi(ghg^{-1}) = \chi_\pi(h)$.*
2. $\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$.
3. $\chi_{\pi^*} = \chi_\pi^{-1}$.
4. $\chi_\pi(e) = \deg(\pi)$
5. $\chi_{\mathrm{Hom}(\pi,\tau)}(g) = \chi_\pi(g^{-1})\chi_\tau(g)$.

Proof. — Exercise. □

Remark 4.4. — It is easier to prove (e) using the tensor product. The rest are straightforward.

Corollary 4.5. —
1. $\chi_\pi(g^{-1}) = \chi_{\pi^*(g)} = \overline{\chi_\pi(g)}$
2. $|\chi_\pi(g)| \leqslant \deg(\pi)$.
3. $\chi_\pi(g) = \deg(\pi) \Leftrightarrow$ *the action of G on V is by scalar multiplication (that is each $\pi(g) = \alpha_g I_V$).*
4. $\chi_\pi = \dim(V) \Leftrightarrow$ *action of G on V is trivial i.e., each $\pi(g) = I$.*

Lemma 4.6. — *Let a finite group G act on a finite set X and let representation obtained by the induced action on $\mathcal{F}(X)$ be denoted by π, i.e.,*

$$(\pi(g)f)(x) = f(g^{-1} \cdot x)$$

for $g \in G, f \in \mathcal{F}(X), x \in X$. The character of the representation π is given by

$$\chi_\pi(g) = |\{x \in X : g \cdot x = x\}|. \tag{4.1}$$

Proof. — Recall that δ_x denotes a basis vector of $\mathcal{F}(X.)$ Note that $[\pi(g)\delta_x](y) = 1$ if and only if $\delta_x(g^{-1} \cdot y) = 1$ if and only if $y = g \cdot x$ if and only if $\delta_{g\cdot x}(y) = 1$. Thus, $\pi(g)\delta_x = \delta_{g\cdot x}$. Note that $\chi_\pi(g) = \sum_{x \in X} \delta_x^*(\pi(g)\delta_x)$, where $\{\delta_x^* : x \in X\}$ is the dual basis of $\{\delta_x : x \in X\}$. But $\delta_x^*(\pi(g)\delta_x) = \delta_x^*(\delta_{g\cdot x}) = 1$ if and only if $g \cdot x = x$. The statement follows. □

Corollary 4.7. — *Let $(\lambda, \mathcal{F}(G))$ and $(\rho, \mathcal{F}(G))$ denote the left and right regular representations of a finite group G. Then, we have*

$$\chi_\lambda(g) = \chi_\rho(g) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Definition 4.8. — Let (π, V) and (τ, W) be representations on a group G. Suppose $T : V \to W$ is a linear map with

$$T(\pi(g)v) = \tau(g)(T(v)), \quad \forall g \in G, \quad v \in V. \tag{4.3}$$

We then say that the linear map T is G-linear (or G-equivariant map). Thus, a linear map $T : V \to W$ is G-equivariant if and only if, for each $g \in G$, the following diagram commutes

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\pi(g)"'] & W \arrow[d, "\tau(g)"] \\ V \arrow[r, "T"] & W \end{tikzcd} .$$

Definition 4.9. — Denote $\mathrm{Hom}_G(\pi, \tau) := \{T : V \to W | T \text{ is } G - \text{equivariant maps}\} \subseteq \mathrm{Hom}_{\mathbb{C}}(V, W)$.

Remark 4.10. — $\mathrm{Hom}_G(\pi, \tau)$ is a complex vector space.

Definition 4.11. — A map $T \in \mathrm{Hom}_G(\pi, \tau)$ is said to be an isomorphism if T is a bijection. Two representations (π, V) and (τ, W) on a group G are said to be *equivalent* if there exists $T \in \mathrm{Hom}_G(\pi, \tau)$ such that T is bijective map.

Remark 4.12. — If T is an isomorphism, then we set from (4.3),

$$\tau(g) = T \circ \pi(g) \circ T^{-1} \tag{4.4}$$

for each $g \in G$.

Example 4.13. — $(\lambda, \mathcal{F}(G))$ and $(\rho, \mathcal{F}(G))$ are equivalent representations.

Example 4.14. — Let $(\tau, \mathbb{C}^2)$ be the representation of S_3 given by $r \mapsto$ and $s \mapsto$. Are τ and π equivalent?

Lemma 4.15. — *Let (π, V) and (τ, W) be representations on a group G and let $T \in$ $Hom_G(\pi, \tau)$. Then*

1. $\mathrm{Ker}(T)$ *is a G-invariant subspace of V.*
2. *$Img(T)$ is a G-invariant subspace of W.*

Proof. — Exercise. □

***Lemma 4.16* (Schur's Lemma)**. — *Let G be a finite group. Suppose that (π, V) and (τ, W) be irreducible finite dimensional representations on G. Then*

1. *If π is not equivalent to τ, then* $\mathrm{Hom}_G(\pi, \tau) = \{0\}$.

2. *If $W = W, \tau = \pi$, and $T \in \mathrm{Hom}_G(\pi, \tau)$, then $T = \lambda I_V$, for some $\lambda \in \mathbb{C}$. In fact, $Hom_G(\pi, \pi) = \{\lambda I | \lambda \in \mathbb{C}\}$.*
3. *If τ is equivalent to π then $\mathrm{Hom}_G(\pi, \tau)$ is a one-dimensional $\mathbb{C}$-vector space.*

Corollary 4.17. — *Suppose G is a finite abelian group, and (π, V) is an irreducible representation of G. Then $\dim(V) = 1$.*

Recall the inner product on $\mathcal{F}(G)$ from (3.1).

***Theorem 4.18* (Basic Formula).** — *Let (π, V) be a finite dimensional representation of a finite group G. Then*

$$\dim(V^G) = \langle \chi_\pi, 1 \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_\pi(g). \tag{4.5}$$

where V^G denotes the subspace of G-fixed vectors.

Proof. — Define $T = \frac{1}{|G|} \sum_{g \in G} \pi(g)$. It is easy to verify the following:

1. $T \in \mathrm{End}(V)$ and $T^2 = T$.
2. $Img(T) = V^G$.
3. $V = \mathrm{Ker}(T) \oplus V^G$
4. Consequently, $\mathrm{tr}(T) = \dim(V^G)$.

Computing trace from the definition of T above and comparing gives us the desired result. □

5. Schur orthogonality relations of characters and consequences

Proposition 5.1. — *Let (π, V_1), (π_2, V_2) be two finite dimensional representations of G a finite group. Then*

$$\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle = \dim(\mathrm{Hom}_G(\pi_1, \pi_2))$$

Proof. — It is easy to check that, $\mathrm{Hom}_G(\pi_1, \pi_2) = (\mathrm{Hom}(\pi_1, \pi_2))^G$. By Theorem 4.18, $\dim((\mathrm{Hom}(\pi_1, \pi_2))^G) = \langle \chi_{(\mathrm{Hom}(\pi_1,\pi_2))^G}, 1 \rangle$. Now, we apply Proposition 4.3 (5) and apply a change of variables $g \mapsto g^{-1}$ to prove the theorem. □

***Theorem 5.2* (Schur Orthogonality Relations).** — *Let π_1, π_2 be two irreducible representations of a finite group G. Then*

$$\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle = \begin{cases} 1 & \text{if } \pi_1 \text{ is equivalent to } \pi_2 \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

Proof. — Follows from Schur's Lemma and the previous proposition. □

Put

$$\begin{aligned} Irr(G) &= \text{set of all irreducible representations of } G \text{ upto equivalence} \\ &= \{[\pi] | \pi \text{ irreducible representation of } G\}. \end{aligned}$$

Corollary 5.3. — *The subset $\{\chi_\pi | \pi \in Irr(G)\}$ of $\mathcal{F}(G)$ is linearly independent.*

Proof. — An orthogonal set in an inner product space is linearly independent. □

Corollary 5.4. — *Let G be a finite group. We have $|Irr(G)| \leqslant |G|$*

Proof. — Since $\dim(\mathcal{F}(G)) = |G|$, the statement follows. □

Theorem 5.5 **(Uniqueness of decomposition into irreducibles)**
Let (π, V) be a finite dimensional representation of G. Let $(\pi, V) = (\tau, W_1) \oplus \cdots \oplus (\tau_r, W_r)$ be a decomposition of V into a direct sum of irreducible representations. Let (ρ, W) be any irreducible representation of G. Let $(\rho, \mathcal{F}(G))$ denote the right regular representation of G. We have

The number of representations (τ_j, W_j) equivalent to the representation (ρ, W)=$\langle \chi_\pi, \chi_\rho \rangle$.

Proof. — Follows from Schur Orthogonality relations. □

Remark 5.6. — Theorem 5.5 says that the decomposition of a finite dimensional representation of a finite group into irreducible ones is independent of the choice of the irreducible components appearing in the decomposition up to isomorphism. Putting this along with Maaschke's theorem, we get an existence and uniqueness theorem for the decomposition of a finite dimensional representation of a finite group into irreducible ones. The reader may compare it with prime factorization of positive integers or a factorization of polynomials in one indeterminate over a filed into irreducible factors.

We now prove a fundamental result in the subject as a consequence to the previous theorem.

Corollary 5.7. — *Suppose (π_1, V_1) and (π_2, V_2) are two finite dimensional representations of a finite group G. Then the following are equivalent:*

1. *(π_1, V_1) is equivalent to (π_2, V_2)*
2. *$\chi_{\pi_1} = \chi_{\pi_2}$*

Proof. — (1) implies (2) is obvious. (2) implies (1) follows from Maaschke's theorem and Theorem 5.5. □

Next, we are going to prove a fundamental theorem of finite group representation theory.

Theorem 5.8. — *Let Irr(G) denote the set of inequivalent irreducible representations of a finite group G.*

1. $\chi_\lambda = \chi_\rho = \sum_{\pi \in Irr(G)} \chi_\pi(e)\chi_\pi = \sum_{\pi \in Irr(G)} \deg(\pi)\chi_\pi$.

2. $|G| = \sum_{\pi \in Irr(G)} (\chi_\pi(e))^2 = \sum_{\pi \in Irr(G)} (\deg(\pi))^2$.

3. $\sum_{\pi \in Irr(G)} \chi_\pi(1)\chi_\pi(g) = 0$ *for each* $g \in G$ *such that* $g \neq e$.

Proof. — First note that, for any $\pi \in Irr(G)$, the number of irreducible component of λ which are equivalent to π equals $\langle \chi_\lambda, \chi_\pi \rangle = \deg(\pi)$. This proves (1).

(2) follows from considering $\langle \chi_\lambda, \chi_\lambda \rangle$ and Schur's orthogonality relations.

(3) follows from (?) □

In the lectures, for lack of time, an outline of the proof that the number of conjugacy classes equals the number of distinct inequivalent irreducible representations of G was given. We state it here for completeness.

***Definition 5.9* (Center of $\mathcal{F}(G)$).** — $Z(\mathcal{F}(G)) = \{f \in \mathcal{F}(G) | f * \phi = \phi * f \, \forall \phi \in \mathcal{F}(G)\}$

***Lemma 5.10*.** — *$f \in Z(\mathcal{F}(G)) \iff f$ is a class function i.e. $f(gxg^{-1}) = f(x)$ for all $g \in G$. In other words, a function on G belongs to $Z(\mathcal{F}(G))$ if and only if f is constant on conjugacy classes.*

Proof. — A function f belongs to $Z(\mathcal{F}(G))$ if and only if $f * \delta_g = \delta_g * f$ for each $g \in G$. The statement follows from this. □

***Theorem 5.11*.** — *Let G be a finite group and let h denote the number of conjugacy classes in G. Then, $|Irr(G)| = h$.*

Proof. — We outline the proof in few simpler steps. The strategy for the proof is that the center $Z(\mathcal{F}(G))$ of the algebra $\mathcal{F}(G)$ is isomorphic to $\mathbb{C}^{|Irr(G)|}$ and that $Z(\mathcal{F}(G))$ has as basis the set of characteristic functions of conjugacy classes of G.

Step 1. — It is straightforward to check that $Z(\mathcal{F}(G))$ is a vector subspace of $\mathcal{F}(G)$ and moreover a $\mathbb{C}$-subalgbera of $\mathcal{F}(G)$, i.e, if $f_1, f_2 \in Z(\mathcal{F}(G))$ then $f_1 * f_2 \in Z(\mathcal{F}(G))$. This follows since $*$ is associative.

Step 2. — We have a map $\Phi : \mathcal{F}(G) \longrightarrow \prod_{\pi \in Irr(G)} \mathrm{End}(V_\pi)$ given by $f \mapsto (\pi(f))_{\pi \in Irr(G))}$ where $\pi(f) = \frac{1}{|G|} \sum_{g \in G} f(g)\pi(g)$. Endow the right hand side with addition, scalar multiplication and composition component-wise to make it a $\mathbb{C}$-algebra. It is straightforward that Φ is a linear map. More importantly,

$$\Phi(f_1 * f_2) = \Phi(f_1) \circ \Phi(f_2)$$

for all $f_1, f_2 \in \mathcal{F}(G)$. This last equality is easy to verify once we show that $\pi(f_1 * f_2) = \pi(f_1) \circ \pi(f_2)$ for all $f_1, f_2 \in \mathcal{F}(G)$, which follows from the definition of $*$ and a change of variables. Thus the map Φ is a homomorphism of $\mathbb{C}$-algebras.

Step 3. — The map Φ is one-one. If $f \in \mathrm{Ker}(\Phi)$, then $\pi(f) = 0$ for each $\pi \in Irr(G)$. In particular, $\lambda(f) = 0$ for the left regular representation λ. Remember that $\lambda(f) = \dfrac{1}{|G|}\displaystyle\sum_{g \in G} f(g)\lambda(g) \in End(\mathcal{F}(G))$. Thus, if $\lambda(f) = 0$, we get in particular that $\lambda(f)(\delta_e) = 0$.

Now, $[\lambda(f)](\delta_e) = 0$ implies $\sum_{g \in G} f(g)\lambda(g)(\delta_e) = 0$ which in turn gives

$$\sum_{g \in G} f(g)\delta_g = 0.$$

Evaluating at a point $x \in G$, we get $f(x) = 0$. As $x \in G$ is arbitrary, we get $f = 0$.

Step 4. — The map Φ is onto as the dimension of the $\mathbb{C}$-vector spaces on both sides are equal by Burnside's formula.

Step 5. — Since the map Φ is an isomorphism of $\mathbb{C}$-algebras, the centers on either side are isomorphic. Thus, $Z(\mathcal{F}(G))$ is isomorphic to the center of $\prod_{\pi \in Irr(G)} \mathrm{End}(V_\pi)$ which is isomorphic to $\mathbb{C}^{|Irr(G)|}$.

Step 6. — Let $\{C_1 \ldots, C_h\}$ denote the conjugacy classes in G. Denote by 1_{C_i} the characteristic function of the conjugacy class C_i i.e., $1_{C_i} : G \to \mathbb{C}$ is defined by

$$1_{C_i}(g) = \begin{cases} 1 & g \in C_i \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\{1_{C_1}, \ldots, 1_{C_h}\}$ forms a linearly independent set in $\mathcal{F}(G)$. Fix representatives $x_i \in C_i$ for each i. Then, by Lemma 5.10, each $f \in Z(\mathcal{F}(G))$ can be written as $f = \sum_{i=1}^{h} f(x_i)1_{Ci}$. Thus, $\{1_{C_1}, \ldots, 1_{C_h}\}$ is a basis of $Z(\mathcal{F}(G))$.

Combining all the steps together, we get

$$|\text{ conjugacy classes in } G| = h = \dim(Z(\mathcal{F}(G))) = \dim(\mathbb{C}^{|Irr(G)|}) = |Irr(G)|.$$

The proof of the theorem is now complete. $\square$

6. Exercises for tutorial sessions

1. Suppose G acts on X. Let $\mathcal{F}(X) := \{f : X \to \mathbb{C}\}$. Show that G acts on $\mathcal{F}(X)$ as follows: $g \in G, x \in X$ $(g \cdot f)(x) = f(g^{-1} \cdot x)$.
2. Suppoe X is a finite set. Consider $(\mathcal{F}(X), +, \cdot)$ where $+$ is pointwise addition and $\cdot$ is scalar multiplication
   (a) Show that $(\mathcal{F}(X), +, \cdot)$ is a $\mathbb{C}$ -vector space. What is the $\dim(\mathcal{F}(X))$?. Can you give a basis of $\mathcal{F}(X)$?

(b) Is the action of G on $\mathcal{F}(X)$ linear ? If so, what is the representation of G on $\mathcal{F}(X)$ obtained via this action?

3. G acts on G via:
   (i) left action: $\lambda(g)x = g \cdot x$
   (ii) right action: $\rho(g)x = x \cdot g^{-1}$
   From Problem 1, G acts on $\mathcal{F}(G)$ via λ and ρ. What are the actions and write down the representations: $\lambda : G \to Aut(\mathcal{F}(G))$ and $\rho : G \to Aut(\mathcal{F}(G))$.
4. Using Problem 3, i.e. λ and ρ, we can define action of $G \times G$ on G; $(g, h) \cdot x = gxh^{-1}$, $g, x, h \in G$. What is the induced action of $G \times G$ on $\mathcal{F}(G)$?
5. Assume that G is finite. On $\mathcal{F}(G)$, define a new operation $*$ as follows: Given $f_1, f_2 \in \mathcal{F}(G)$, define

$$f_1 * f_2 : G \to \mathbb{C} \quad \text{by}$$
$$(f_1 * f_2)(x) = \frac{1}{|G|} \sum_{g \in G} f_1(g) f_2(g^{-1}x)$$

   (a) Show that $*$ is associative.
   (b) Show that $(\mathcal{F}(G), +, \cdot, *)$ is a $\mathbb{C}$-algebra.
   (c) Does $(\mathcal{F}(G), +, *)$ have an identity element?

6. Show that the representation $(\pi, \mathbb{C}^2)$ of S_3 we defined in the lecture is irreducible. What about the two dimensional representation of D_4 we defined in the lecture?

7. Assume G is finite. On $\mathcal{F}(G)$, define a form: $f_1, f_2 \in \mathcal{F}(G)$, define

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

   (a) Show that $\langle , \rangle$ is a G-invariant inner product on $\mathcal{F}(G)$.
   (b) Let W denote the subspace of constant functions on G. Show that W is G-invariant.
   (c) What is a G-invariant complement of W in $\mathcal{F}(G)$?

8. Suppose (π, V) is a finite dimensional representation of a group G. Define

$$V^G = \{v \in V : \pi(g)v = v \text{ for all } g \in G\}.$$

   Verify that V^G is a G-invariant subspace. This subspace is called the space of G-fixed vectors.

9. Suppose (π, V) is a representation of a finite group G.
   (a) Let $v \in V$ be a non-zero vector. What is the smallest G-invariant subspace of V containing v?
   (b) If (π, V) is irreducible, what can you say about the subspace you obtained in (a)?
   (c) If (π, V) is irreducible, what can you conclude about $\dim(V)$?

10. Given a representation (π, V) of a finite group G, define a function $\chi_\pi : G \to \mathbb{C}$ as follows:

$$\chi_\pi(g) = tr[\pi(g)]$$

for each $g \in G$. This invariant of the representation is called the character of the representation (π, V).

Verify the following:

(a) If $g, h \in G$, show that $\chi_\pi(ghg^{-1}) = \chi_\pi(h)$.
(b) $\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$.
(c) $\chi_{\pi^*} = \chi_\pi^{-1}$.
(d) $\chi_\pi(e) = \dim(V)$
(e) $\chi_{\mathrm{Hom}(\pi,\tau)}(g) = \chi_\pi(g^{-1})\chi_\tau(g)$.(It is easier to do this using tensor product).

11. Let a finite group G act on a finite set X and let representation obtained by the induced action on $\mathcal{F}(X)$ be denoted by π, i.e.,

$$(\pi(g)f)(x) = f(g^{-1} \cdot x)$$

for $g \in G, f \in \mathcal{F}(X), x \in X$.

(a) Calculate the character of the representation π.
(b) If $X = G$ and $\pi = \lambda$, the left regular action, what is χ_λ?
(c) If $X = G$ and $\pi = \rho$, the right regular action, what is χ_ρ?

12. Can you give an example of a one-dimensional representation of the group $GL_2(\mathbb{F}_p)$?(Hint: determinant). What are all the one dimensional representations of the group $GL_2(\mathbb{F}_p)$?
13. What is the center of
    (a) the group S_n.
    (b) the group D_n.
    (c) the group A_n.
    (d) the group $GL_2(\mathbb{F}_p)$.
    (e) the group $SL_2(\mathbb{F}_p)$.
14. What are the conjugacy classes in
    (a) the group S_n.
    (b) the group D_n.
    (c) the group A_n.
    (d) the group $GL_2(\mathbb{F}_p)$.
    (e) the group $SL_2(\mathbb{F}_p)$.
15. Compute the character tables of the groups:
    (a) the group $\mathbb{Z}/n\mathbb{Z}$.
    (b) the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.
    (c) the group $\mathbb{Z}_n \times \mathbb{Z}_n$.
    (d) the group S_3.
    (e) the group D_4.
    (f) the group A_4.

References

J- P. Serre, *Finite groups - an introduction*, International Press of Boston, 2016.