Assignment sheet
The Fourier transform

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1. Let $f \in \mathfrak{F}_a$ with $a > 0$, then $\forall n \in \mathbb{N}$, show that $f^{(n)} \in \mathfrak{F}_b,\ 0 \leq b < a$.

2. Suppose $f \in \mathfrak{F}_a,\ 0 < b < a, x, R \in \mathbb{R}$ and V is the line joining $R - ib,\ R + ib$, then find $\lim\limits_{R\to\infty} \int\limits_V \frac{f(z)}{z-x} dz,\ \lim\limits_{R\to-\infty} \int\limits_V \frac{f(z)}{z-x} dz.$

3. Compute the Fourier transform of $(i)\ e^{-\pi x^2}$, $(ii)\ \frac{1}{\pi}\frac{y}{y^2+x^2}$, $(iii)\ \operatorname{sech}(\pi x)$.

4. Suppose f is continuous and of moderate decrease, and $\hat{f}(\xi) = 0,\ \forall \xi \in \mathbb{R}$. Show that $f = 0$.

   <u>Hint</u>: Fix $t \in \mathbb{R}$. Set $A(z) = \int\limits_{-\infty}^{t} f(x) e^{-2\pi i z(x-t)} dx,\ B(z) = -\int\limits_{t}^{\infty} f(x) e^{-2\pi i z(x-t)} dx,$ $F(z) = \begin{cases} A(z), & \operatorname{Im}(z) \geq 0, \\ B(z), & \operatorname{Im}(z) < 0. \end{cases}$ Show that F is entire, bounded function. Then show that $F \equiv 0$. From this deduce that $\int\limits_{-\infty}^{t} f(x)dx = 0,\ \forall t$, and conclude $f \equiv 0$.

5. Verify the Fourier inversion formula for the functions given in Exercise 3.

6. If $A > 0,\ B \in \mathbb{R}$, then find $\int\limits_0^{\infty} e^{-(A+iB)x} dx.$

7. If $g : \mathbb{R} \to \mathbb{C}$ be continuous, then show that $f(z) = \int_{-n}^{n} g(t) e^{2\pi i t z} dt$ is entire.

8. Let f has moderate decrease, $a \in \mathbb{R},\ t > 0$. Then prove that following:

   (i) If $g_1(x) = f(x)e^{2\pi i a x}$, then $\hat{g}_1(\xi) = \hat{f}(\xi - a)$,

   (ii) If $g_2(x) = f(x-a)$, then $\hat{g}_2(\xi) = \hat{f}(\xi)e^{-2\pi i a \xi}$,

   (iii) If $g_3(x) = f(\frac{x}{t})$, then $\hat{g}_3(\xi) = t\hat{f}(t\xi)$.

9. Prove the following identities for $a \in \mathbb{R},\ t > 0$.

   $(i)\ \sum\limits_{n=-\infty}^{\infty} e^{-\pi t(n+a)^2} = \sum\limits_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{t}} \frac{e^{2\pi i a n}}{\sqrt{t}},$

   $(ii)\ \sum\limits_{n=-\infty}^{\infty} \frac{e^{-2\pi i a n}}{\cosh(\frac{n\pi}{t})} = \sum\limits_{n=-\infty}^{\infty} \frac{t}{\cosh(\pi t(n+a))},$

   $(iii)\ \frac{1}{\pi} \sum\limits_{n=-\infty}^{\infty} \frac{a}{a^2+n^2} = \sum\limits_{n=-\infty}^{\infty} e^{-2\pi a|n|} = \coth(\pi a).$

10. Prove or disprove: The function $f(z) = e^{z^2}$ satisfies the maximum priciple in any unbounded domain.

11. Let S be a sector whose vertex is the origin, and forming an angle of $\frac{\pi}{\beta}$. Let F be a holomorphic function in S that is continuous on the closer of S, so that $|F(z)| \leq 1$, $z \in \partial S$. Moreover, let there exist $C, a > 0$, $\alpha \in (0, \beta)$ such that $|F(z)| \leq Ce^{a|z|^\alpha}$, $\forall z \in S$. Prove that $|F(z)| \leq 1$, $\forall z \in S$.
Hint: Use the technique employed the proof of Phragmén-Lindelöf theorem.