AFS: Complex Analysis, module 3, Exercises

1. Path homotopy is an equivalence relation.

2. If α and β are curves in Ω from z_0 to z_1 and z_1 to z_2, respectively, define $\alpha * \beta$ from z_0 to z_2 by

$$\alpha * \beta(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2}, \\ \beta(2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

If α_0 and α_1 are path homotopic and β_0 and β_1 are path homotopic, then $\alpha_0 * \beta_0$ is path homotopic to $\alpha_1 * \beta_1$.

3. Define the reverse $\overleftarrow{\alpha}$ of α by $\overleftarrow{\alpha}(t) = \alpha(1-t)$. If α is path homotopic to β, then $\overleftarrow{\alpha}$ is path homotopic to $\overleftarrow{\beta}$ and $\alpha * \overleftarrow{\alpha}$ and $\overleftarrow{\alpha} * \alpha$ are path homotopic to the constant paths ϵ_{z_0} and ϵ_{z_1}, respectively.

4. Ω is simply connected iff every closed curve at $z_0 \in \Omega$ is null homotopic, i.e. is path homotopic to the constant loop at z_0.

5. A star-like domain is simply connected. Give an example of a simply connected domain that is not star-like.

6. $\gamma_0(t) = e^{2\pi \imath t}, 0 \leq t \leq 1$ is homotopic (not path homotopic) to the constant path $\gamma_1(t) \equiv 1$ in $\mathbb{C}^*$.

5. Which of the following are simply connected?

   a) $H \setminus [0, \imath]$ where $H = \{z = x + \imath y : y > 0\}$.
   b) $H \setminus [\imath, 99\imath]$
   c) $\mathbb{C} \setminus [0, \infty)$
   d) $\mathbb{C} \setminus [-2022, 2022]$.

6. There is no continuous function $\theta : \mathbb{C}^* \to \mathbb{R}$ such that $z = |z|e^{\imath\theta(z)}$ for $z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

7. If u is harmonic, then $g = u_x - \imath u_y$ is holomorphic.

8. In a simply connected domain, every harmonic function has a harmonic conjugate.

9. Find a harmonic function in $\mathbb{C}^*$ having no harmonic conjugate.

10. If f is holomorphic and never vanishes on Ω, then $\log|f|$ is harmonic.