Tutorial: Free Groups, Generators and Relations

1. Prove or Disprove: The free group on two generators is isomorphic to product of two infinite cyclic groups.

2. Can every finite group G be presented by a finite set of generators and a finite set of relations?

3. Let F be a free group on $\{x, y\}$. Prove that the two elements $u = x^2$ and $v = y^3$ generate a subgroup of F isomorphic to the free group on u, v.

4. Let F be a free group on $\{x, y\}$. Prove that the three elements $u = x^2$, $v = y^2$ and $z = xy$ generate a subgroup isomorphic to the free group on u, v, z.

5. A subgroup H of a group G is characteristic if it is carried to itself by all automorphisms of G (if $\phi \in Aut(G)$, then $\phi(H) \leq H$).

   - Prove that every characteristic subgroup is normal and that the center $Z(G)$ is a characteristic subgroup.
   - Determine the normal subgroups and the characteristic subgroups of the quaternion group.
   - Prove that the subgroup H generated by all the elements in G of order n is characteristic.

6. The commutator subgroup C of a group G is the smallest that contains all commutators(for $a, b \in G$, define the commutator $[a, b] = a^{-1}b^{-1}ab$). Prove that the commutator subgroup is a characteristic subgroup and that G/C is an abelian group.