CHAPTER 6 OF ARTIN'S ALGEBRA: "MORE GROUP THEORY"

NOTES AND TUTORIAL SHEET 3

Abstract. Semi-direct product: this is a useful general construction; it helps in particular in producing a larger group from two known smaller ones, when one of the smaller ones can act on the other "by group automorphisms"; Groups of order 12.

(1) Suppose that N is a group and there is an action of a group H on N "by group automorphisms." What this means is the following: we have, first of all, an action map $H \times N \to N$ satisfying the usual axioms for a group action; let us denote the result of h in H acting on n in N by ${}^h n$ (as will presently become clear, this notation is much better than the usual hn in this case); the extra condition here (that of the action being by group automorphisms) means that[1] the map $n \mapsto {}^h n$ is a group homomorphism from N to N for every fixed h: that is ${}^h(nn') = {}^h n {}^h n'$.

  (a) We may define a group structure on the Cartesian product $N \times H$ of the sets N and H by:

$$(n,h)(n',h') := (n{}^h n', hh') \quad \text{for } n \text{ in } N \text{ and } h \text{ in } H \tag{1}$$

  The resulting group is called the semi-direct product of N and H and denoted $N \rtimes H$.

  (b) The map $n \mapsto (n,1)$ for $n \in N$ defines an injective group homomorphism of N onto its image $N \times \{1\}$ in $N \rtimes H$. Similarly, the map $h \mapsto (1,h)$ for $h \in H$ defines an injective group homomorphism of H onto its image $\{1\} \times H$ in $N \rtimes H$. We identify N and H with their images under these maps and say that N and H are subgroups of $N \rtimes H$.

  (c) N is a normal subgroup of $N \rtimes H$ (which justifies the notation $N \rtimes H$). We have ${}^h n = hnh^{-1}$ for h in H and n in N, where the left side denotes the action (originally given) of h on n and the right side is interpreted as an expression in the group $N \rtimes H$.

  (d) In case the original action is the trivial one, that is to say ${}^h n = n$ for all h and n, then the semi-direct product reduces to the direct product.

(2) Let N and H be subgroups of a larger ambient group and suppose that N is normal (or, more generally, that H normalises N). Let $NH := \{nh \mid n \in N, h \in H\}$.

  (a) NH is a subgroup. (Hint: NH is non-empty since 1 belongs to it; it is closed under multiplication and taking inverses. We have $nhn'h' = (n(hn'h^{-1}))(hh') = (nn'')(hh')$, where $n'' = hnh^{-1}$, so NH is closed under multiplication; and $(nh)^{-1} = h^{-1}n^{-1} = (h^{-1}n^{-1}h)h^{-1} = n'^{-1}h^{-1}$, where $n' = h^{-1}nh$, so NH is closed under taking inverses.)

  (b) The conjugation action of H on N, namely that defined by ${}^h n = hnh^{-1}$, is an action by group automorphisms, and we may form the semi-direct product $N \rtimes H$.

  (c) The map $N \rtimes H \to NH$ defined by $(n,h) \mapsto nh$ is a surjective group homomorphism. It is an isomorphism if and only if $N \cap H = \{1\}$.

[1] Alternatively, and equivalently, we may say that the group homomorphism $H \to \mathfrak{S}_N$ from H to the group $\mathfrak{S}_N$ of bijections of N that is defined by the action has image landing in the subgroup $\mathrm{Aut}(N)$ of $\mathfrak{S}_N$ consisting of those bijections from N to N that are also group homomorphisms: $H \to \mathrm{Aut}(N) \subseteq \mathfrak{S}_N$.

(3) Let N be an abelian group and $H = \{1, s\}$ the group with two elements ($s^2 = 1$).
   (a) Then H acts on N by group automorphisms: ${}^s n := n^{-1}$; and we may form the semi-direct product $N \rtimes H$.
   (b) In case $N = \{1, r, r^2, \ldots, r^{n-1}\}$ is the cyclic group of order n with generator r, then $N \rtimes H$ is nothing but the familiar dihedral group D_n (of order $2n$).

(4) In this item, we will construct all groups, up to isomorphism, of order 12. We will see that there are five different ones, all of which can be realized as semi-direct products. Let G be a group of order 12. Let N be Sylow-2 subgroup of G and K a Sylow-3 subgroup.
   (a) $|N| = 4$ and $|K| = 3$; there are two possibilities for N: it is either cyclic or the Klein four group; as for K, it is cyclic.
   (b) Show that at least one of N and K is normal. Thus we may form at least one of the semi-direct products $N \rtimes K$ or $K \rtimes N$, depending upon which group is normal. Since $N \cap K = \{1\}$, it follows that $G = NK$ and from item (2c) that G is isomorphic to any semi-direct product that can be formed.
   (c) Suppose that N is normal. Then the conjugation action of K on N is by group automorphisms and $G \simeq N \rtimes K$. We have two cases:
      (i) In the case N is cyclic, there being no automorphisms of order three of N, the action is trivial and we have $G = N \times K$ (the cyclic group of order 12).
      (ii) In the case N is the Klein four group, then we have two further cases: in case the action is trivial, G is the abelian group $C_2 \times C_2 \times C_3$; in case the action is non-trivial, then $G \simeq A_4$.
   (d) Suppose that K is normal and N is not normal (the case when N is normal being taken care of in the previous item). Then the conjugation action of N on K is by group automorphisms and $G = N \ltimes K$. We have two cases:
      (i) In case N is the Klein four group, G is the dihedral group D_4.
      (ii) In case N is the cyclic group, G is the group that is described in Artin by generators and relations as follows:

$$\left\langle x, y \,|\, x^4 = 1, y^3 = 1, xy = y^2 x \right\rangle$$