CHAPTER 6 OF ARTIN'S ALGEBRA: "MORE GROUP THEORY"

NOTES AND TUTORIAL SHEET 2A

Abstract. Induced actions: for instance, on the power set, on Cartesian products. Analysing these actions is useful to deduce information about the original situation.

If a group acts on a set, then it acts naturally on natural constructions that one can make from that set.[1] These actions are called induced actions. For example, given a set X with the action of a group G, there is a natural induced action of G on:

- the power set of X: let $gU := \{gu \mid u \in U\}$, for U a subset of X and g an element of G.
- the Cartesian product $X \times X$: let $g(x_1, x_2) := (gx_1, gx_2)$ for (x_1, x_2) in $X \times X$ and g in G. This is called the diagonal action. A similar diagonal action can be defined on $X^3 := X \times X \times X$, X^4, X^5, ...

(1) Let G be a group, X a G-set, and consider the induced action of G on the power set of X. This induced action preserves cardinalities: for a subset U of X, its image gU under the action of any element g in G has the same cardinality as U, so subsets of X of a fixed cardinality form a G-stable subset of its power set.

(2) Let $G = \mathfrak{S}_n$ be the group of bijections from the set $X := \{1, \dots, n\}$ to itself.
   (a) Determine the orbits for the (induced) action of G on the power set of X.
   (b) Determine the orbits for the (induced) action of G on $X \times X$, $X \times X \times X$, ...

(3) Let V a vector space and $GL(V)$ the group of invertible linear transformations on V. Recall that $GL(V)$ is the "symmetry group" of V and we have the "defining action" $GL(V) \times V \to V$ of $GL(V)$ on V. Consider the induced action of $GL(V)$ on the power set $\mathscr{P}(V)$ of V.
   (a) For g in $GL(V)$ and W a linear subspace of V, the image gW (of W under the induced action of g in G) is also a linear subspace of the same dimension as W. For an integer $k \geqslant 0$, let $\mathbb{G}(k, V)$ denote the collection of all linear subspaces of V of dimension k.[2] The preceding remark shows that $\mathbb{G}(k, V)$ is a $GL(V)$-stable subset of $\mathscr{P}(V)$. The action of $GL(V)$ on $\mathbb{G}(k, V)$ is transitive and so we may identify $\mathbb{G}(k, V)$ with the coset space $GL(V)/H$, where H is the stabiliser of any k-dimensional subspace W of V.
   (b) The case $k = 1$ of the previous item is especially important. The collection $\mathbb{G}(1, V)$ of 1-dimensional subspaces of V (also called lines) is called the projective space over V.
   (c) Now let V be a vector space of finite dimension n over a finite field $\mathbb{F}_q$ of cardinality q. Use the counting formula (and knowledge of the cardinality of $GL(V) \simeq GL(n, \mathbb{F}_q)$) to calculate the cardinality of the projective space over V. Do the same for other $\mathbb{G}(k, V)$.

[1] More generally, if a group acts on a collection of sets, then it acts on any natural combination of them. For example, given two sets X and Y on which a group G acts, G also acts on the set of all maps from X to Y: we define $(gf)(x) := f(g^{-1}x)$, for g in G, x in X, and $f : X \to Y$.

[2] The symbol $\mathbb{G}$ stands for Grassmannian, in honour of the mathematician Grassmann.

(4) Consider the left regular action of a group G on itself and the induced action on the power set of G.
   (a) If H is a subgroup of G, then the left cosets of H form a G-orbit in the power set of G.
   (b) Let U be a subset of G and H a subgroup of G. The stabiliser G_U of U (for the induced G-action on the subsets of G) contains H if and only if U is a union of (some) right cosets of H. In particular, U is always the union of (some) right cosets of G_U. If U is finite, then $|G_U|$ divides $|U|$. And if $|U|$ is corpime to $|G|$, then G_U is trivial and the orbit of U has order equal to $|G|$.

(5) Using the observations in the item (4) above, let us determine the orbits of the G-action on the set of unordered pairs of elements of G (induced from the left regular action) in the case $G = \mathfrak{S}_3$. There are $\binom{6}{2} = 15$ such pairs. We have three subgroups of order 2 in G: namely, $\{1, (12)\}$, $\{1, (13)\}$, and $\{1, (23)\}$. By item (4a), the set of cosets of each of these subgroups forms a G-orbit. Each of these orbits contains 3 pairs, and so their union accounts for 9 pairs. The remaining 6 pairs are all in a single G-orbit: the stabiliser of any of those pairs is the trivial subgroup (see item (4b)).

(6) The dihedral group D_4 of symmetries of the square acts on the vertices of the square (and by this means can be realized as a subgroup of $\mathfrak{S}_4$). What are the orbits for the resulting action of D_4 on the subsets of the vertices?

(7) Consider the group A_4 of rotational symmetries of the regular tetrahedron. What are the orbits for the resulting action on subsets of the vertices? (Hint: We may think of this action as the restriction to A_4 of the natural action of $\mathfrak{S}_4$ on subsets of $\{1, 2, 3, 4\}$.)

(8) Consider the "octohedral" group O of the 24 rotational symmetries of the cube. Consider the induced action of O on (unordered) pairs of vertices of the cube. There are $\binom{8}{2} = 28$ such pairs of vertices, and they get divided into three orbits. The pairs in each of these orbits have the following neat geometrical description:
   (a) the vertices in the pair are joined by an edge (there are as many such pairs as edges in the cube, and so there are 12 of them).
   (b) the vertices are opposite vertices of one of the faces of the cube (there are two such pairs for every face, and so there are 12 such pairs of vertices, the six faces of the cube contributing 2 such pairs each).
   (c) the vertices are opposite vertices of the cube, that is, there is no face of the cube containing both of them (there are 4 such pairs).

(9) Consider the conjugation of a group G on itself and the induced action on the power set of G. Let H be a subgroup.
   (a) The image gHg^{-1} of H under the action of g in G is also a subgroup. Subgroups of this form (that is, those that are equal to gHg^{-1} for some g in G) are called <u>conjugates</u> of H. Thus the orbit of H is the set of conjugates of H.
   (b) The stabiliser of H is just its normaliser $N_G(H) := \{g \in G | gHg^{-1} = H\}$. Evidently $H \subseteq N_G(H)$. Moreover H is a normal subgroup of its normaliser and the normaliser is the largest subgroup of G that contains H as a normal subgroup. In particular, $N_G(H) = G$ if and only if H is normal in G.
   (c) In case G is finite, we obtain the following equation from the counting formula: $|G| = |\{\text{conjugates of } H\}| \cdot |N_G(H)|$.