Topology
Tutorial Problems: Week-2

Exercise 1
Let X be a topological space and $A \subset X$. Assume that for each $x \in A$ there exists an open set U such that $U \subset A$. Show that A must be open.

Exercise 2
Let X be a nonempty set and let $\tau = \{U \subset X : X \setminus U \text{ is infinite or } \phi \text{ or } X\}$. Is τ a topology on X?

Exercise 3
Show that $\mathbb{R}^n$ is second countable. Is the space $C[a, b]$ second countable?

Exercise 4
Let X be a topological space and Y a subspace of X. Assume that $A \subset Y$. Show that if A is closed in Y and Y is closed in X, then A must be closed in X.

Exercise 5
Let Y be a subspace of a topological space X and let $A \subset Y$. Show that the closure of A as a subset of Y is $\overline{A} \cap Y$.

Exercise 6
Let X and Y be topological spaces. If $A \subset X$ and $B \subset Y$ are closed sets, then show that $A \times B$ must be closed in $X \times Y$.

Exercise 7
Let (X, d) be a metric space. Define $d' : X \times X \to \mathbb{R}$ by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that the metric d' induces the same topology on X as d.

Exercise 8
Let X and Y be topological spaces and $f : X \to Y$ be a mapping. Show that f is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$. Give an example where this inclusion is strict.

Exercise 9
(Pasting lemma) Let $\{A_i\}$ be a collection of open sets in X such that $\cup A_i = X$. Assume that $f : X \to Y$ is such that every $f_i := f|_{A_i} : A_i \to Y$ is continuous. Show that f must be continuous. Is this true if instead we assume that each A_i is closed in X?

Exercise 10

Let X and Y be topological spaces. Define the first projection map $\pi_1 : X \times Y \to X$ as $\pi_1(x, y) = x$. Show that the map is open. Is this map closed as well?

Exercise 11

Let X be a topological space and $\{A_i\}$ be a collection of subsets of X. Show that

$$\cup_i \overline{A_i} \subset \overline{\cup_i A_i}.$$

Give an example to show that this inclusion can be strict. What if there are only finitely many subsets A_i?

Exercise 12

Is it true in general that $\overline{A \cap B} = \overline{A} \cap \overline{B}$?

Exercise 13

Let X and Y be topological spaces and that $A \subset X$ and $B \subset Y$. Show that

$$\overline{A \times B} = \overline{A} \times \overline{B}.$$

Exercise 14

Prove the following.

(i) Every metric space must be a Hausdorff space.

(ii) Any subspace of Hausdorff space must be a Hausdorff space.

(ii) Any product of Hausdorff spaces must be a Hausdorff space.

Exercise 15

Show that a topological space X is Hausdorff if and only if its diagonal

$$\Delta := \{(x, x) : x \in X\}$$

is a closed subset of $X \times X$.

Exercise 16

Assume that X and Y are topological spaces and Y is Hausdorff. Suppose that $f, g : X \to Y$ are continuous functions. Show that the set $\{x \in X : f(x) = g(x)\}$ is closed in X.

Exercise 17

Give an example to show that the existence of embeddings $f : X \to Y$ and $g : Y \to X$ does not imply that the topological spaces X and Y are homeomorphic.

Exercise 18

Give an example to show that a continuous bijective map $f : X \to Y$ need not be a homeomorphism.

Exercise 19

Let (X, d) be a compact metric space and $f : X \to X$ be a map that satisfies the condition

$$d(f(x), f(y)) = d(x, y)$$

for all $x, y \in X$. Show that f must be a homeomorphism.

Exercise 20

Let $A \subset (X, d)$ be totally bounded. Show that $\overline{A}$ is also totally bounded.

Exercise 21

(i) Give an example of a metric space which is not separable.

(ii) Prove that every compact metric space must be separable.

Exercise 22

Let X be a Hausdorff space and $A,\ B$ be disjoint compact subsets of X. Show that there exist disjoint open sets $U,\ V \subset X$ such that $A \subset U$ and $B \subset V$.

Exercise 23

Assume X is compact and Y is Hausdorff. Show that any continuous map $f : X \to Y$ must be a closed map.

Exercise 24

Show that if Y is compact, then the first projection $\pi_1 : X \times Y \to X$ is a closed map.

Exercise 25

Let Y be compact and Hausdorff. Show that a map $f : X \to Y$ is continuous if and if its graph

$$\{(x, y) \in X \times Y : y = f(x)\}$$

is closed in $X \times Y$.

Exercise 26

Let (X, d) be a metric space, A nonempty closed subset and B nonempty compact subset of X with $A \cap B = \phi$. Show that $d(A, B) := inf\{d(x, y) : x \in A \ and \ y \in B\}$ must be positive.

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