Topology

Tutorial Problems: Week-I

Exercise 1

Let $f : X \to Y$ be a function. Define a relation $\sim$ on X as follows:

$$x_1 \sim x_2 \text{ } \textit{if } f(x_1) = f(x_2).$$

Show that $\sim$ is an equivalence relation on X. What are the corresponding equivalence classes?

Exercise 2

Assume that the set X is countably infinite. Show that the n-fold cartesian product X^n is also countably infinite.

(In particular, $\mathbb{Q}^n$ is countably infinite.)

Exercise 3

Let X be any set. Show that there does not exist a bijection between X and its power set $P(X)$.

Exercise 4

(i) Show that every countably infinite set is equivalent to a proper subset of itself.

(ii) Show that any infinite set is equivalent to a proper subset of itself.

Exercise 5

Prove that the set $[0, 1) \times [0, 1)$ is equivalent to the set $[0, 1)$.

(Hint: Use Schroeder-Bernstein Theorem.)

Exercise 6

A complex number is said to be algebraic if it is a zero of a polynomial with integer coefficients. Otherwise it is said to be a trancendental number. Show that the set of all algebraic numbers is countably infinite. As a consequence, deduce that there exists a trancendental real number.

Exercise 7

Let (X, d) be a metric space. Define $d' : X \times X \to \mathbb{R}$ by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that d' is a metric on X. Further, show that a subset $U \subset X$ is open in (X, d) if and only if it is open in (X, d').

Exercise 8

Let (X_1, d_1) and (X_2, d_2) be metric spaces. Define $d : X_1 \times X_2 \to \mathbb{R}$ and $d' : (X_1 \times X_2) \times (X_1 \times X_2) \to \mathbb{R}$ as follows:

$$d((x_1, x_2), (x'_1, x'_2)) = \max\{d_1(x_1, x'_1), d_2(x_2, x'_2)\},$$
$$d'((x_1, x_2), (x'_1, x'_2)) = d_1(x_1, x'_1) + d_2(x_2, x'_2).$$

Show that d_1 and d_2 are metrics on $X_1 \times X_2$.

Exercise 9

Give a counterexample for each of the following statements.

(i) The intersection of any collection of open sets must be open.

(ii) The union of any collection of closed sets must be open.

(iii) In any metric space (X, d) we have

$$\overline{B(x_0, r)} = \{x \in X : d(x, x_0) \leq r\}$$

for every $r > 0$.

Exercise 10

Let (X, d) be a metric space and (Y, d) be a subspace of (X, d). Assume that $Z \subset Y$. Prove that Z is open in Y if and only if there exists an open set O in X such that $Z = O \cap Y$.

Exercise 11

Let (X, d) be any metric space and $x_0 \in X$. Define a function $f : X \to \mathbb{R}$ by

$$f(x) = d(x, x_0).$$

Is the function f continuous on X?

Exercise 12

Let (X, d) be a metric space. Assume that $x_0 \in X$ and $r > 0$. Prove that the set

$$\{x \in X : d(x, x_0) \leq r\}$$

is a closed subset of X.

Exercise 13

Let (X, d) be a metric space.

(i) Assume that $x \in X$, and $A \subset X$ is a closed set such which does not contain x. Show that we can find disjoint open subsets U and V of X such that $x \in U$ and $A \subset V$.

(ii) Assume that A and B are disjoint closed sets in X. Show that we can find disjoint open subsets U and V of X such that $A \subset U$ and $B \subset V$.

Exercise 14

Let (X, d) be a metric space and $A \subset X$. Show that

$$\overline{A} = \{x \in X : d(x, A) = 0\}.$$

Exercise 15

Let (X, d) be a metric space and U is an open subset of X. Assume that $A \subset X$. Show that $U \cap A = \phi$ if and only if $U \cap \overline{A} = \phi$.

Exercise 16

Let $A = \mathbb{Q} \cap [0, 1] \subset \mathbb{R}$. What is the boundary of A?

Exercise 17

Let (X, d) be a metric space. Assume that $\{x_n\}$ and $\{y_n\}$ are sequences in X converging to x and y. Show that the sequence $\{d(x_n, y_n)\}$ converges to $d(x, y)$.

Exercise 18

Show that a Cauchy sequence converges if and only if it has a convergent subsequence.

Exercise 19

Prove that the metric space $C[a, b]$ is complete. Is this space separable? Is it second countable?

Exercise 20

Let X and Y be metric spaces and $f : X \to Y$ be a mapping. Show that f is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$. Give an example where this inclusion is strict.

Exercise 21

For any metric space (X, d), let $C(X)$ denote the set of all bounded continuous real-valued function on X. For $f, g \in X$, set

$$d(f, g) = sup_{x\in X}|f(x) - g(x)|.$$

Prove that (X, d) is a complete metric space.

Exercise 22

Let (X, d) be a complete metric space and Y be a subspace. Show that Y is complete if and only if it is closed.

Exercise 23

(Completion) Let (X, d) be a metric space. Fix any point x_0 in X.

(i) For any $x \in X$, define the map $f_x : X \to \mathbb{R}$ by $f_x(y) = d(y, x) - d(y, x_0)$. Show that the map f_x is continuous and bounded.

(ii) By (i) we obtain a map $F : (X, d) \to (C(X), d')$ given by $F(x) = f_x$. Show that F is an isometry, i.e.

$$d'(F(x), F(y)) = d(x, y).$$

(iii) Define completion X^* of X to be the closure of $F(X)$ in $C(X)$. Show that X^* is a complete metric space which contains an isometric image of X that is dense in X^*.

****