PROBLEM SET FOR THE 1ST WEEK (COMPLEX ANALYSIS)

Fundamental structures of $\mathbb{C}$:

(1) As we know $\mathbb{C}$ has a pair of binary operations on it namely, addition and multiplication satisfying certain basic axioms, out of which every algebraic property/theorem of complex numbers can in principle (even if the chains of logic involved in the derivation are quite long) be derived; these axioms are essentially associativity, existence of an additive and multiplicative identity, existence of additive inverse for all $z \in \mathbb{C}$, existence of multiplicative inverse for every $z \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and the distributivity of multiplication over addition. All of these are enjoyed by real numbers as well, which also has an order structure in it. Prove that it is impossible to linearly/totally order the complex numbers by an order relation which is compatible with its algebraic structure (unlike the reals). To make this assertion precise, we have the following definitions:
Let X be any (abstract) set. A relation $R \subset X \times X$, which we shall denote $\preceq$ is called a *preorder* if it satisfies the following axioms:
(i) Reflexivity: $x \preceq x$ holds for all $x \in X$,
(ii) Transitivity: if $x \preceq y$ and $y \preceq z$ for some $x, y, z \in X$, then $x \preceq z$ also holds.
The relation $\preceq$ is called a *partial order* if it also satisfies:
(iii) Anti-symmetry axiom: if $x \preceq y$ and $y \preceq x$ then $x = y$.
A partial order is called a *total/linear order* if every pair of elements are comparable i.e., given any pair $x_1, x_2 \in X$, we *must* either $x_1 \preceq x_2$ or $x_2 \preceq x_1$. The strict-ordering associated to a given preorder $\preceq$ on X is the relation defined (and denoted) by $x_1 \prec x_2$ if $x_1 \preceq x_2$ but $x_1 \neq x_2$.

Examples/Exercises:
(a) Pick any set S and define $A \preceq B$ if $A \subset B$; then (verify that) this gives a partial order on the power set of S which is not a total order.
(b) For $z, w \in \mathbb{C}$ define $z \preceq w$ if $|z| \leq |w|$; this gives a pre-order which is not a partial order.
(c) For $z, w \in \mathbb{C}$ define a total order relation by comparing their real and imaginary parts as follows. Declare $z \preceq w$ if $\Re(z_1) \prec \Re(z_2)$ or if $\Re(z_1) = \Re(z_2)$ *and* $\Im(z_1) \preceq \Im(z_2)$. This is called the lexicographic/dictionary order and it gives a total order on $\mathbb{C}$ but it is not compatible with the algebraic operations of $\mathbb{C}$.
(d) For $z, w \in \mathbb{C}$ define $z \preceq w$ if $\Re(z_1) \preceq \Re(z_2)$ *and* $\Im(z_1) \preceq \Im(z_2)$ (this is essentially what is known as the 'product order' on $\mathbb{R}^2$). Then this is a partial order which is not a total order, which is compatible with addition but not with multiplication.

Finally, prove that there does not exist a total ordering on $\mathbb{C}$ which is compatible with the field operations in the following sense. Whenever $z_1 \preceq z_2$ and $0 \preceq z_3$, we

have: $z_3 z_1 \preceq z_3 z_2$. If we also have another pair of complex numbers w_1, w_2 satisfying $w_1 \preceq w_2$, then we have $z_1 + w_1 \preceq z_2 + w_2$. Prove then, that it is impossible to put an order relation $\preceq$ on $\mathbb{C}$ satisfying the above conditions.
Remark: The example in (b) above namely, $z \preceq w$ if $|z| \preceq |w|$ is compatible with multiplication in this sense (but not with addition). However, more importantly as already noted above, it does not give even a partial order, let alone a total ordering.

(2) Show that $d(z, w) = |z - w|$ defines a distance function which makes $\mathbb{C}$ into a complete metric space. Next, verify the reverse triangle inequality $|w - z| \geq \big||w| - |z|\big|$ for all $z, w \in \mathbb{C}$ with equality iff either z and w are positive multiples of each other. Conclude that the absolute value function $z \to |z|$ is continuous on $\mathbb{C}$.

(3) Show that $\mathbb{C}$ defined as the set of all expressions of the form $a + ib$ where a, b are real numbers, with addition and multiplication defined by

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$
$$(a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

is 'isomorphic' to

$$\mathcal{M} := \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \ : \ a, b \in \mathbb{R} \right\}$$

where by 'isomorphic', we mean the existence of a bijective mapping $F : \mathbb{C} \to \mathcal{M}$ satisfying: $F(z + w) = F(z) + F(w)$ and $F(zw) = F(z)F(w)$ wherein the operations on the right-hand-side of this pair of equations are matrix-addition and matrix-multiplication respectively; such a map F is called a field isomorphism.

(4) Show that $\mathbb{F}[t]$, the polynomial ring in the single variable t with coefficients from a field $\mathbb{F}$ is a PID (principal ideal domain), thereby the rings $\mathbb{R}[x]$ and $\mathbb{C}[z]$ are PIDs. Show that the element $x^2 + 1$ is irreducible in $\mathbb{R}[x]$ and thereby is a prime element in $\mathbb{R}[x]$ and therefore the quotient ring $\mathbb{R}[x]/\langle x^2 + 1\rangle$ is an integral domain. Show that the ideal $\langle x^2 + 1\rangle$ is a maximal ideal in $\mathbb{R}[x]$ and thereby the quotient ring $\mathbb{R}[x]/\langle x^2 + 1\rangle$ is actually field and finally show that this field is isomorphic to the field of complex numbers.

(5) Prove that the conjugation mapping is the only field automorphism of $\mathbb{C}$ which maps $\mathbb{R}$ into itself and is different from the identity map. (Hint: First prove that a field automorphism f of $\mathbb{C}$ with $f(\mathbb{R}) \subset \mathbb{R}$ must fix $\mathbb{R}$ pointwise (i.e., $f(x) = x$ for each $x \in \mathbb{R}$) by showing that any such f must be order preserving).

'Real' applications of complex algebra and calculus:

(6) Application of complex algebra to deriving trigonometric identities:
(i) Prove/Recall that for any complex number $\neq 1$, we have for each $n \in \mathbb{N}$ the following equality of a polynomial and a rational function:

$$1 + z + z^2 + \ldots + z^n = \frac{z^{n+1} - 1}{z - 1}$$

Use this to derive the following trigonometric identities valid for all real numbers θ which are not an integer multiples of 2π:

$$1 + \cos\theta + \cos 2\theta + \ldots + \cos n\theta = \frac{1}{2} + \frac{\sin\big((2n+1)\theta/2\big)}{2\sin\theta/2}$$

and

$$\sin\theta + \sin 2\theta + \ldots + \sin n\theta = \frac{\sin(n\frac{\theta}{2})\sin\big((n+1)\frac{\theta}{2}\big)}{\sin\frac{\theta}{2}}.$$

(ii) Derive a 'real'-identity for $\cos^4\theta$ in terms of multiples of θ i.e., an expression of $\cos^4\theta$ which is expressed as a (finite) linear combination of functions of the form $\cos n\theta$ where $n \in \mathbb{Z}$ and which is valid at-once for *all real* values of θ.

Remark: These identities arise in (the basic) theory of Fourier series.

(7) (Application to finding indefinite integrals of functions of a real variable – later in the course, applications of complex calculus to challenging definite integrals of functions of a real variable will be discussed): Compute

$$\int e^{3x}\cos 2x dx$$

i.e., determine (upto additive constants) an antiderivative of $e^{3x}\cos 2x$. It's possible to do this by integrating by parts twice; to do this with much fewer computations, use the relation that exists between the integrand to a complex-valued function of a real variable (i.e., view the integrand as the real-part of a complex-valued function of a real variable).

(8) For which values of θ does the sequence $e^{in\theta}$ converge? Needless to say, you must justify your answer. Indeed, your justification/analysis must help you to show that both the limits $\lim_{n\to\infty}\cos n\theta$ and $\lim_{n\to\infty}\sin n\theta$ fail to exist whenever θ is not an integer multiple of π.

CR (Cauchy – Riemann) equations and 'Wirtinger derivatives'.

(9) Recall the partial differential operators (called the 'Wirtinger derivative' operators):

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Express the CR equations as a single equation in terms of the above operators. Show that if f is holomorphic at z_0, then

$$\frac{\partial f}{\partial \overline{z}} = 0, \text{ and } f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0)$$

where we write f in-terms of its real and imaginary parts as $f = u + iv$. Further next, if we write f as $F(x, y)$, then show that: F is differentiable in the sense of real variables and $\det(J_F(x_0, y_0) = |f'(z_0)|^2$.

(10) Show that the CR-equations can be written in polar coordinates in the form:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations, to show that the (principal branch of the complex) logarithm function defined by

$$\log(z) = \log r + i\theta$$

where $z = re^{i\theta}$ with $-\pi < \theta < \pi$, is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

(11) Show that the standard Laplace operator in 2 real variables namely, $\partial^2/\partial x^2 + \partial^2/\partial y^2$ can also be expressed as:

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$$

Deduce that if f is holomorphic on an open set U, then the real and imaginary parts of f are *harmonic* functions i.e., their Laplacian vanishes identically.

Basic Complex Line integrals:

(12) Evaluate the line-integrals

$$\int_\gamma z^n dz$$

for all integers n. Here γ is any circle centered at the origin, with the positive (counterclockwise) orientation. Do the same, for circles γ not containing the origin in its 'inside'.

(13) Show that if $|a| < r < |b|$, then:

$$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b},$$

where γ denotes the circle centered at the origin, of radius r, with positive orientation.

Anti-derivatives/primitives

(14) Suppose f is continuous on a region/domain Ω. Prove that any two primitives of f (if they exist) differ by a constant.

Goursat's theorem, Cauchy's theorem, Cauchy Integral formula and its consequences

(15) Suppose f is continuously *complex* differentiable on a domain $\Omega \subset \mathbb{C}$. Suppose T is a triangle whose interior/inside is also contained within Ω. Show using Green's

theorem that

$$\int_\gamma f(z)dz = 0.$$

(<u>Remark</u>: This provides a proof of Goursat's theorem under the additional assumption that f' is continuous).

(16) Let Ω be a domain in $\mathbb{C}$ and T is a triangle whose interior/inside is also contained within Ω. Suppose that f is a function holomorphic in Ω, except possibly at a point w inside T. Prove that if f is bounded near w, then:

$$\int_T f(z)dz = 0.$$

(17) Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Let d denote the diameter of the image, given precisely by

$$d := \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$$

Show that this diameter satisfies $2|f'(0)| \leq d$.
(Hint: For all $0 < r < 1$, we have

$$2f'(0) = \frac{1}{2\pi i}\int_{|\zeta|=r}\left(\frac{f(\zeta) - f(-\zeta)}{\zeta^2}d\zeta\right).$$

(18) Suppose that Ω is a bounded domain in $\mathbb{C}$ and $\varphi : \Omega \to \Omega$, a holomorphic map with the properties that, there exists $z_0 \in \Omega$ such that:

$$\varphi(z_0) = z_0, \ \text{ and } \varphi'(z_0) = 1$$

Prove that φ must be a linear map.

(19) Let f be a holomorphic function on the disc D_{R_0} centered at the origin, of radius R_0. Prove that for all $0 < R < R_0$ and $|z| < R$, we have

$$f(z) = \frac{1}{2\pi i}\int_0^{2\pi} f(Re^{i\varphi})\mathsf{Re}\left(\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z}\right)d\varphi.$$

(20) Let u be a twice continuously differentiable function on the standard unit disc $\mathbb{D}$, which is harmonic. Prove that there exists a holomorphic function f on the unit disc such that $\mathrm{Re}(f) = u$. Also show that the imaginary part of f is uniquely defined upto an additive (real) constant.
(Hint: Recall from an earlier problem that $f'(z) = 2\partial u/\partial z$. Therefore, let $g(z) = 2\partial u/\partial z$ and prove that g is holomorphic. Why can one find F with $F' = g$? Prove that $\mathrm{Re}(f)$ differs from u by a real constant).

(21) From the above result, deduce the Poisson integral representation formula from the Cauchy Integral Formula: if u is harmonic in the unit disc and continuous on

its closure, then if $z = re^{i\theta}$, one has

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \varphi) u(\varphi) d\varphi,$$

where $P_r(\varphi)$ is the Poisson kernel for the unit disc give by

$$P_r(\gamma) = \frac{1 - r^2}{1 - 2r\cos\gamma + r^2}.$$

(22) Suppose f is a non-vanishing continuous function on $\overline{\mathbb{D}}$ which is holomorphic in $\mathbb{D}$. Prove that if

$$|f(z)| = 1, \quad \text{whenever } |z| = 1$$

then f is constant.