LINEAR GROUPS - PROBLEM SHEETS COMBINED (3^{rd} WEEK)

1. Problems on Linear Groups

(1) State and prove Cayley's Theorem (using group action). Prove that every finite group of order n is isomorphic to a closed subgroup of the orthogonal group $O(n)$.

(2) Prove that the following defines a group action of the group $GL_n(\mathbb{R})$ on $M_n(\mathbb{R})$

$$A \to (P^t)^{-1}A(P^{-1}).$$

Find the orbit and stabilizer of various choice of matrices in $M_n(C)$.

(3) Let $\alpha \in [-1, 1]$. Define $C_\alpha = \{M \in SU_2(\mathbb{C}) : \text{ Trace}(M) = 2\alpha\}$. Prove that every conjugacy class in $SU_2(\mathbb{C})$ is of this form C_α for some α.

(4) Prove that the group $SU_2(\mathbb{C})$ is homeomorphic to the unit 3-sphere in $\mathbb{R}^4$. Further, describe/identify the latitudes and longitudes of the 3-sphere under this homeomorphism.

(5) Prove that the center of the group $SU_2(\mathbb{C})$ is equal $\{\pm I\}$.

(6) Consider the set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Give an example of a topological space X and a subspace Y such that $A \subsetneq Y$ and A is closed in Y.

(7) Let Y be a subspace of X. Prove that a set $A \subseteq Y$ is closed in Y if and only if $A = Y \cap C$ where C is closed in X.

(8) Show that the group $O_1(\mathbb{R})$ is isomorphic to the cyclic group of order 2 and that $SO_2(\mathbb{R})$ is isomorphic as a topological group to the circle group S^1. Conclude that $O_2(\mathbb{R})$ is homeomorphic as a topological space to $SO_2(\mathbb{R}) \times O_1(\mathbb{R})$, but that as a group these objects are not isomorphic to each other.

(9) Prove that $\frac{O(n)}{O(n-1)}$ is homeomorphic to S^{n-1}.

(10) Prove that $\frac{SO(n)}{SO(n-1)}$ is homeomorphic to S^{n-1}.

(11) Prove that $O_n(\mathbb{R})$ has exactly two connected components.

(12) Show that $GL_n(\mathbb{F})$ is a subgroup of $SL_{n+1}(\mathbb{R})$.

(13) Prove that $GL_n(\mathbb{Q})$ is a subgroup of $GL_n(\mathbb{R})$ but it is not closed.

(14) Prove that the exponential of a matrix is well-defined.

(15) Prove the Lie product formula for two complex matrices X and Y:

$$e^{X+Y} = \lim_{m\to\infty} (e^{(\frac{X}{m})}(e^{(\frac{Y}{m})}))^m.$$

(16) Prove that, for a complex matrix X

$$\det(e^X) = e^{\text{tr}(X)}$$

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Please turn over...

(17) Let G be a matrix Lie group. Define its Lie algebra to be the set

$$\mathfrak{g} := \{X \in M_n(\mathbb{C}) : e^{tX} \in G \ \forall t \in G\}.$$

Prove that $\mathfrak{g}$ satisfies all the conditions of an abstract Lie algebra for a suitable choice of the bracket.

(18) Compute the dimensions of the following groups.
  (a) $SU_n(\mathbb{C})$
  (b) $SO_n(C)$
  (c) $SP_{2n}((R)$
  (d) $O_{3,1}$.

(19) Let a be an irrational real number and define

$$G = \{\begin{bmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{bmatrix} : t \in \mathbb{R}\}$$

Prove that G is not a matrix Lie group. Also, prove that G is a subgroup of $GL_2(\mathbb{C})$.

(20) The Lie algebra of $U(n)$ consists of all complex matrices X satisfying $X^* = X$ and $\operatorname{tr}(X) = 0$.

(21) The Lie algebra of orthogonal group $O(n)$ consists of all real matrices X satisfying $X^{\text{tr}} = X$.

(22) Prove that the Lie algebra of the Heisenberg group

$$H = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

is the space of all matrices of the form

$$\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix},$$

where $a, b, c \in \mathbb{R}$.

Please turn over...

Groups	Closed	Bounded	Compact	Connected	Components	Simply connected	Fundamental Group	
$GL_n(\mathbb{C})$								
$SL_n(\mathbb{C})$								
$GLn_(\mathbb{R})$								
$SL_n(\mathbb{R})$								
$O_n(\mathbb{R})$								
$SO_n(\mathbb{R})$								
$On(\mathbb{C})$								
$SO_n(\mathbb{C})$								
$U_n(\mathbb{C})$								
$SU_n(\mathbb{C})$								
$Sp_n(\mathbb{R})$								
$Sp_n(\mathbb{C})$								
$O(n, 1)$								
$SO(n, 1)$								
$O(n, k)$								
$SO(n, k)$								
Heisenberg								
$E(n)$								
$P(n, 1)$								

Fill in the blanks by *Yes*/*No* with the justification.