AIS IN COMMUTATIVE ALGEBRA, CMI, DEC 2015

TUTORIAL PROBLEMS

1. 2015-12-28

(1.a) Show that $X_1X_2X_3 + Y_1Y_2Y_3 \notin \mathbb{F}_2[X_i + Y_i, X_iY_i, X_iY_j + X_jY_i, 1 \leq i \leq 3]$ using the $\mathbb{N}^3$ grading where $\deg X_i = \deg Y_i$ is the i standard vector in $\mathbb{N}^3$.

(1.b) Let $R = \Bbbk[X_1, X_2, X_3, Y_1, Y_2, Y_3]$ where $\operatorname{char} \Bbbk \neq 2$. Compute $R^{\langle\sigma\rangle}$ where $\sigma(X_i) = Y_i$ and $\sigma(Y_i) = X_i$ for every i.

(1.c) K. R. Nagarajan's example. Let $\Bbbk = \mathbb{F}_2(a_1, b_1, a_2, b_2, \ldots,)$ and $R = \Bbbk[[X, Y]]$. Define $p_n = a_nX + b_nY$, $\sigma(X) = X$, $\sigma(Y) = Y$, $\sigma(a_n) = a_n + Yp_{n+1}$ and $\sigma(b_n) = b_n + Xp_{n+1}$. Let $G = \langle\sigma\rangle$. Observe that $p_n \in R^G$ for every n. Show that $p_{n+1} \notin (p_1, \ldots, p_n)R^G$ for every n following the steps below:

(a) If $f \in \mathbb{F}_2[a_1, b_1, a_2, b_2, \ldots,]$, then $\sigma(f) \equiv f \mod \mathfrak{m}^2$ where $\mathfrak{m} = (X, Y)R$.

(b) If $f \in \Bbbk$, then $\sigma(f) \equiv f \mod \mathfrak{m}^2$.

(c) For $r \in R$, set $\overline{r}$ to be the constant term of r. If $r \in R^G$, then $\sigma(\overline{r}) \equiv \overline{r} \mod (X^2, Y^2)R$.

(d) If $p_{n+1} = \sum_{k=1}^{n} r_kp_k$ with $r_k \in R^G$, then $a_{n+1} = \sum_{k=1}^{n} \overline{r_k}a_k$.

(e) Use the above result to show that

$$a_{n+2} = \sum_{k=1}^{n} \overline{r_k}a_{k+1}$$

$$a_{n+3} = \sum_{k=1}^{n} \overline{r_k}a_{k+2}$$

$$\cdots$$

(f) Show that this is not possible.

(1.d) Let $R = \mathbb{F}_3[X, Y, Z]$. Is there an R^{S_3}-linear projection $R \longrightarrow R^{S_3}$? Is there an R^{A_3}-linear projection $R \longrightarrow R^{A_3}$?

2. 2015-12-29

(2.a) Let X be a 2×3 matrix of variables, and $R = \Bbbk[X]/I_2(X)$. Compute $\operatorname{depth} R$, $\dim R$, $H_R(t)$ (the Hilbert series) and $\deg_R(R)$.

(2.b) Let $G \subseteq \mathrm{GL}_n(\Bbbk)$ consist of diagonal matrices. Show that $\Bbbk[X_1, \ldots, X_n]^G$ is generated by monomials in $X_1, \ldots, X_n$.

(2.c) Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ and $\chi : G \longrightarrow \mathbb{C}^*$ a character. Compute $\deg_{R^G} R^G_\chi$.

(2.d) Compute R^G where $R = \mathbb{F}_p[X, Y]$ and G is the subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$ generated by

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

`problem:diagonal`

(2.e) Fix $d \in \mathbb{N}$ and $\zeta = e^{\frac{2\pi\imath}{d}}$.

(a) Let G be the subgroup of $\mathrm{GL}_n(\mathbb{C})$ generated by ζI_n. Let $R = \mathbb{C}[X_1, \ldots, X_n]$. Compute R^G and $H_{R^G}(t)$.

(b) Evaluate

$$\sum_{n=1}^{d-1} \frac{1}{(1-\zeta^n)(1-\overline{\zeta^n})}.$$

3. 2015-12-30

(3.a) Let $R = \Bbbk[x^4, x^3y, xy^3, y^4]$ generated by elements of degree 1. Compute $H_R(t)$. Is R Cohen-Macaulay?

(3.b) Let $\phi : R \longrightarrow S$ be a surjective $\Bbbk$-algebra morphism. Let $\{s_i\}$ be $\Bbbk$-basis for S and let $\{r_i \in R\}$ be such that $\phi(r_j) = s_j$. Suppose that $I \subseteq \ker\phi$. If each element of R is congruent to an element in the $\Bbbk$-span of $\{r_i\}$ modulo I, then $I = \ker\phi$. problem:twobythree

(3.c) Let $\phi : R := \Bbbk[u, v, w, x, y, z] \longrightarrow \Bbbk[ar, br, cr, as, bs, cs] =: S$ be $\Bbbk$-linear with $\phi(u) = ar, \cdots, \phi(z) = cs$.
  (a) Show that $\ker\phi = (uy - vx, vz - wy, wx - uz)$
  (b) Check that $\Bbbk[u, v - x, w - y, z] \subseteq S$ is a Noether normalization and that S is free over this normalization.

(3.d) Are the following CM?
  (a) $\Bbbk[x, y, z]/(xy, yz, zx)$
  (b) $\Bbbk[w, x, y, z]/(wx, wy, zx, zy)$

(3.e) Let $G \subseteq \mathrm{GL}_3(\mathbb{C})$ be the subgroup generated by

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \imath \end{bmatrix}$$

Compute R^G and $H_{R^G}(t)$. What is the number of pseudo-reflections in G?

4. 2015-12-31

(4.a) Disprove the following statement: Let G is a finite subgroup of $\mathrm{GL}_n(\Bbbk)$, where $\operatorname{char}\Bbbk = 0$. If

$$H_{R^G}(t) = \frac{\sum t^{c_j}}{\prod(1 - t_i^k)}$$

then there exists a homogeneous system of parameters $f_1, \ldots, f_n$ with $\deg f_i = k_i$ and a basis $\{a_j\}$ with $\deg a_j = c_j$ such that $R^G = \oplus a_j \Bbbk[f_1, \ldots, f_n]$.

(4.b) Let p be a prime number. Find $G \subseteq \mathrm{SL}_2(\mathbb{F}_p)$ such that $\deg H_{R^G}(t) \neq -2$.

(4.c) Let G be a finite subgroup of $\mathrm{GL}_n(\Bbbk)$, where $\operatorname{char}\Bbbk = 0$. Suppose that $\Bbbk[f_1, \ldots, f_n]$ is a Noether normalization. Prove that R^G is generated over $\Bbbk$ by elements with degree at most $\max\{\deg f_1, \ldots, \deg f_n, (\sum_i \deg f_i - n)\}$.

(4.d) Show that the bound above is sharp for A_n and better than the Noether bound.

(4.e) Suppose that $\Bbbk[f_1, \ldots, f_n] \subseteq R^G$ is a Noether normalization. Then $|G|$ divides $\prod_i \deg f_i$.

5. 2016-01-01

(5.a) Are the following rings Gorenstein?
  (a) R^G from (2.e).
  (b) Is $\Bbbk[ar, br, cr, as, bs, cs, at, bt, ct]$ Gorenstein? (It is CM.)
  (c) R^G from (3.c).

(5.b) Determine R^G for the subgroup G of $\mathrm{SL}_2(\mathbb{C})$ generated by

$$\begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where $\zeta = e^{\frac{2\pi\imath}{2n}}$.