(KNR) Tutorial sheet 4

1. $G = \mathfrak{S}^n \circlearrowright V = K\langle 1, \dots, n\rangle$ (defining repn.). Determine $\dfrac{K[V]}{K[V]^G_+ K[V]}$ as a G-module.

2. G fin. gp. K field. Char $K \nmid |G|$. V ~~f.d.~~ K-linear repn of G. Show that $V^G \neq 0$ iff $(V^*)^G \neq 0$. (Do this more generally if G is linearly reductive over K.)

3. Show that M-Spec$(K[V]^G) = V/G$ where K = alg. closed field, G finite gp, V f.d. K-linear rep of G. (Proof: Clear that the map M-Spec$(K[V]) \to$ M-Spec$(K[V]^G)$ is surjective and factors through V/G. To show that $V/G \to$ M-Spec$(K[V]^G)$ is injective, enough to ~~find $f \in K[V]^G$~~ ~~s/t (given in~~ show: given two distinct orbits O_1 and O_2 of G on V, $\exists\, f \in K[V]^G$ s/t $f(O_1) = 0$ and $f(O_2) \neq 0$. Let I_1 & I_2 be the ideals in $K[V]$ of O_1 and O_2 respectively. Since $I_1 + I_2 = K[V]$, $\exists\, f_1 \in I_1$ & $f_1' \in I_2$ s/t $f_1 + f_1' = 1$. So f vanishes on O_1 and evaluates to 1 at every point of O_2. Take $f = \prod_{g \in G} {}^g f_1$. This does the job. (In fact, one could just take f_1 to be zero on some point of O_1 and nowhere zero on O_2.))

4. Let G finite gp, K field. V a f.d. K-linear rep of G. Let W be a f.d. G-stable subspace of $K[V]$. ~~Show~~ observe that $V \to W^*$ given by $v \mapsto ev_v : f \mapsto f(v)$ is G-equivariant polynomial map.

5. G finite gp, K field. Let $G \circlearrowright G$ by left multiplication. Let V be the linearization of this action. Let W be the 1-dimensional space spanned by $\sum_{g \in G} g$. Show that W has a G-complement in V if and only if Char $K \nmid |G|$. (Compare with problem #2.)