KNR Tutorial Sheet 2

1. The dimension of the graded piece of degree d in a polynomial ring of n variables $K[x_1, \ldots, x_n]$ is $\binom{n+d-1}{n-1}$. The dimension of the space of polynomials of degree $\leq d$ in n variables is $\binom{n+d}{d}$. Conclude that $K[V]^G$ is generated as an algebra over K with at most $\binom{|G|+\dim V}{\dim V}$ generators (G finite gp, K char 0, V f.d./K G-rep).

2. $H \trianglelefteq G^{\text{finite gp}}$. Then $\pi_{G,H} : U^H \to U^H$ given by $u \mapsto \frac{1}{[G:H]} \sum_{g \in G/H} g u$ is an idempotent G-linear projection with image U^G. (U is any G-linear repn).

3. The following are equivalent for a f.d. G-linear repn:
   a. Every G-invariant subspace admits a G-invariant complement
   b. It is a sum of irreducible G-invariant subspaces
   c. It is a direct sum of G-invariant irreducible subspaces.

   If these equivalent conditions hold, the representation is said to be <u>semisimple</u>.

4. Using the transfer map with $H=\{1\}$ or otherwise, show that ~~any~~ condition ⓐ of 3 above holds for any $G^{\text{finite gp}}$-rep over a field K such that $\operatorname{char} K \nmid |G|$. (This ~~is what is~~ any f.d. G-linear repn over K is semisimple. This is what is meant by "G is a linearly reductive gp" over a field K such that $\operatorname{char} K \nmid |G|$.)

5. Let V & W be G-linear reps with V f.d. Then $V^* \otimes W \simeq \operatorname{Hom}(V, W)$ naturally as G-modules.

6. (General version of 3). A be a ring (not necessarily commutative with identity). The following are equivalent for an A-module M: ⓐ every submodule N admits a complement ⓑ M is a ~~direct~~ sum of simple submodules ⓒ M is a direct sum of simple submodules. Such a module M is called <u>semisimple</u>.

A bit involved. Refer to Jacobson.